IEEE Catalog Number
Part number: CFP21S92-ART
ISBN: 978-1-6654-1854-6

acit.wunu.edu.ua
ACIT

2021 11th International Conference on

Advanced Computer Information Technologies

Deggendorf, GERMANY, 15-17 September 2021

IEEE Germany Section
WEST UKRAINIAN NATIONAL UNIVERSITY
DEGGENDORF INSTITUTE OF TECHNOLOGY
Jihočeská univerzita v Českých Budějovicích
University of South Bohemia in České Budějovice

CONTENTS

SECTION 1
Mathematical Models of Objects and Processes

Approximation of Systems with Delay and Algorithms for Modeling Their Stability

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***Abstract*—In this paper, the schemes of approximation of linear systems with delays by special systems of ordinary differential equations are considered and the connections between their solutions are investigated. Numerical algorithms for studying the stability of linear stationary systems with delay are constructed and their coefficient regions of stability are modeled.**

Keywords*— systems with delay, quasi-polynomial, stability, approximation schemes, upper limit of stability, region of stability.

I. Introduction

In mathematical modeling of physical and technical processes, the evolution of which depends on prehistory, we arrive at differential equations with a delay. With the help of such equations it was possible to identify and describe new effects and phenomena in physics, biology, technology [1].

An important task for differential-functional equations is to construct and substantiate finding approximate solutions, since there are currently no universal methods for finding their precise solutions. Of particular interest are studies that allow the use of methods of the theory of ordinary differential equations for the analysis of delay differential equations.

Schemes for approximating differential-difference equations by special schemes of ordinary differential equations are proposed in the works [2,3]. Further research was found in I. M. Cherevko, L. A. Piddubna, O. V. Matwiy's works [4,5] in various functional spaces.

The study of approximation of linear stationary systems with a delay allowed us to construct algorithms for approximate detection of nonasymptotic roots of quasi-polynomials. Using these algorithms, a method for modelling the stability of solutions of linear systems with a delay is developed, as well as constructive computational algorithms for constructing coefficient regions of stability of linear systems with many delays [6].

The obtained algorithms for finding non-asymptotic quasipolynomial roots and constructing regions of stability of linear delay differential equations can be used to study applied problems of optimal control, modeling of dynamic processes in economics, ecology and others.

II. Approximation Schemes

A. *Approximating of differential-difference equations*

Consider the Cauchy problem for a delayed differential equation

$$\frac{dx}{dt} = F(t, x(t), x(t-\tau)), \qquad (1)$$

$$x(t) = \varphi(t), t \in [t_0 - \tau, t_0], \qquad (2)$$

where $x \in R^n, \tau > 0, t_0 \in R, F(t,u,v)$ is a continuous function.

Equation (1) corresponds to an approximating system of ordinary differential equations

$$\frac{dt_0}{dt} = F(t, z_0, z_m),$$
$$\frac{dz_j}{dt} = \frac{m}{\tau}(t_{j-1}(t) - z_j(t)), j = \overline{1,m}, \qquad (3)$$

$$z_j(t_0) = \varphi(t_0 - \frac{j\tau}{m}), j = \overline{0,m}. \qquad (4)$$

Theorem 1 *[2,4]. If the solution of problem (1) - (2) satisfies the Lipschitz condition on* $[t_0 - \tau, T]$, *then* $\left|x(t - \frac{j\tau}{m}) - z_j(t)\right| \le \frac{K\tau}{\sqrt{m}}, t_0 \in [t_0, T], K > 0.$

If the solution of the problem (1) –(2) $x(t) \in ([t_0 - \tau, T]$, *then* $\left|x(t - \frac{j\tau}{m}) - z_j(t)\right| \le \beta(\omega(x, \frac{\tau}{m})), j = \overline{0,m}, t \in [t_0, T]$, *where* $\beta(\delta) \to 0$ *npu* $\delta \to 0$, $\omega(x, \frac{\tau}{m})$ – *the continuity modulus of the function* $x(t)$ *on* $[t_0 - \tau, T]$.

Note that according to Cantor's theorem on uniform continuity $\omega(x, \frac{\tau}{m}) \to 0$ when $m \to 0$. Therefore, for large m the solution of the Cauchy problem for the system of ordinary differential equations (3) - (4) approximates the solution of the initial problem for the delay equation (1) - (2).

B. Approximating of differential-functional equations

Let us consider the initial problem for functional differential equation

$$\frac{dx(t)}{dt} = L(t, x_t) + f(t, x_t), \quad t \in [0, T], \tag{5}$$

$$x_0 = \varphi,$$

where T – positive constant,

$$\varphi \in C; \; L(t,\varphi) = \sum_{k=0}^{p} A_k(t)\varphi(-\tau_k) + \int_{-\tau}^{0} D(t,\theta)\varphi(\theta)d\theta$$

– a linear functional which is most commonly applied as $A_k(t)$, $k = \overline{0, p}$, $n \times n$ – matrix functions having continuous components if $t \in [0,T]$, $D(t,\theta)$, $n \times n$ – matrix function with continuous components $d_{ij}(t,\theta)$ on the set of variables function on $[0,T] \times [-\tau, 0]$, $0 < \tau_0 < \tau_1 < \ldots < \tau_p = \tau$; $f = (f_1, \ldots, f_n) : \mathbb{R} \times C \to \mathbb{R}^n$ is a continuous function which satisfies the Lipschitz condition $|f(t,\varphi_1) - f(t,\varphi_2)| \le L|\varphi_1 - \varphi_2|$, $L > 0$.

We shall put a system of ordinary differential equations in conformity with (5)

$$\frac{dz_0(t)}{dt} = \sum_{i=0}^{p} A_i(t) z_{l_i}(t) + \frac{\tau}{m}\sum_{i=0}^{m-1} D(t, -\frac{\tau(m-k)}{m}) z_{m-i}(t) + f(t, \sum_{i=1}^{m} z_i \chi_i), \tag{6}$$

$$\frac{dz_j(t)}{dt} = \frac{m}{\tau}(z_{j-1}(t) - z_j(t)), \; j = \overline{1,m}, \; t \in [0,T],$$

$$z_j(0) = \varphi(-\frac{\tau j}{m}), \; j = \overline{0,m}, \tag{7}$$

where $l_i = \left[\frac{m\tau_i}{\tau}\right]$, $\chi_i = \begin{cases} 1, \; \theta \in \left[-\frac{i\tau}{m}, -\frac{(i-1)\tau}{m}\right], \\ 0, \; otherwise. \end{cases}$

Theorem 2 [7-8]. *If $A_k(t)$, $k = \overline{0, p}$, $D(t,\theta)$ – continuous matrix functions, $t \in [0,T]$, $\theta \in [-\tau, 0]$, function $f(t,\varphi)$ is continuous and holds the Lipschitz condition for then the following correlations for φ, the initial problem solution (5) and the Cauchy problem solution (6)-(7) are true*

$$\| x(t - \frac{\tau j}{m}) - z_j(t) \| \to 0, j = \overline{0,m}, \; t \in [0,T] \;\; as \;\; m \to \infty.$$

Example1. Consider the initial problem

$$\frac{dx(t)}{dt} = -1.5x(t) - 1.25x(t-1) + x(t)\sin x(t), \; t \in [0,3],$$

$$x(t) = 10t + 1, \; t \in [-1, 0].$$

Corresponding approximating system of ordinary differential equations has the form

$$\frac{dz_0(t)}{dt} = -1.5z_0(t) - 1.25z_m(t) + z_0(t)\sin z_0(t),$$

$$\frac{dz_j(t)}{dt} = m(z_{m-j}(t) - z_j(t)), \; j = \overline{1,m},$$

$$z_j(0) = \frac{-10j}{m} + 1, \; j = \overline{0,m}.$$

Figure 1 contains results of numerical experiments of calculating $z_0(t)$ when m = 8, 16, 32, using the first order Gear difference scheme. To compare the obtained values we provide a solution $x(t)$ of the initial problem, found in [9] using the block method of the fourth order.

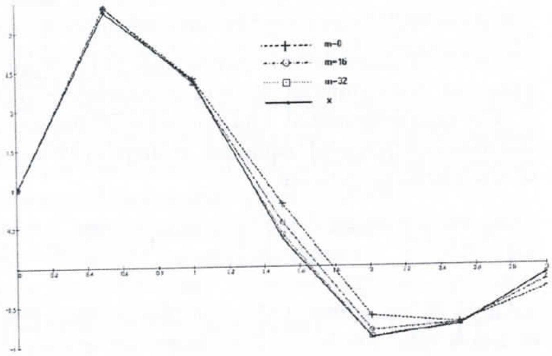

Fig.1

III. Approximation Nonasymptotic Roots of Quasipolynomial

A linear system of differential equations with many delays

$$\frac{dx(t)}{dt} = \sum_{i=0}^{k} A_i x(t - \tau_i), \tag{8}$$

where $x \in R^n$, $A_i, i = \overline{1,k}$, -$n \times n$ are constant matrices, $0 = \tau_0 < \tau_1 < \ldots < \tau_k = \tau$.

The quasipolynomial for system (8) is of the form

$$\Phi(\lambda) = \det(\lambda E - \sum_{i=1}^{k} A_i e^{-\lambda \tau_i}). \tag{9}$$

For linear systems of ordinary differential equations, the characteristic equation is a polynomial. In this case, a number of criteria are known to describe the location of the roots of the characteristic equation, and modern software contains built-in functions for finding them.

For quasi-polynomials of systems of DDE calculation of roots or characteristics of their placement is a difficult task. Let us consider its solution with the help of the proposed schemes of approximation of differential-difference equations.

Following the approximation scheme (3)-(4), we associate with equation (8) the system of ordinary differential equations

$$\frac{dz_0(t)}{dt} = \sum_{i=0}^{k} A_i z_{l_i}(t), \; l_i = \left[\frac{\tau_i m}{\tau}\right], \tag{10}$$

$$\frac{dz_i(t)}{dt} = \mu\left[z_{i-1}(t) - z_i(t)\right], \; i = \overline{1,m}, \; \mu = \frac{m}{\tau}, \; m \in N.$$

For the characteristic equation of approximate system (10), we have the following equality

$$\Psi_m(\lambda) = \det(\lambda E - \sum_{i=0}^{k} A_i (\frac{\mu}{\mu + \lambda})^{l_i})(\mu + \lambda)^{mn}. \tag{11}$$

Let us establish a connection between a quasipolynomial (9) and a characteristic polynomial (11).

Lema 1 [4]. *For fixed $\lambda \in Z$ the sequence of functions*

$$H_m(\lambda) = \frac{\Psi_m(\lambda)}{(\mu+\lambda)^{mn}} \quad (12)$$

converges as $m \to \infty$ to quasipolynomial (9).

We can use this property to find an approximate nonasymptotic root of quasi-polynomia (9). Since zeros of functions $\Psi_m(\lambda)$ and $H_m(\lambda)$ converge by (12), we can take the zeros of characteristic polynomial (11) as approximate values on nonasymptotic zeros of quasipolinomial (9).

For quasipolynomial approximation of the second order difference-differential equation system (10) we obtain a characteristic polynomial

$$\Psi_m(\lambda) = s^{2\cdot m+2}\cdot\left(\frac{m}{\tau}\right)^2 + s^{2\cdot m+1}(-a_{11}\frac{m}{\tau} - a_{22}\frac{m}{\tau} -$$

$$-2\cdot\left(\frac{m}{\tau}\right)^2) + s^{2\cdot m}(a_{11}a_{22} + a_{11}\frac{m}{\tau} + a_{22}\frac{m}{\tau} + \left(\frac{m}{\tau}\right)^2 -$$

$$-a_{21}a_{12}) + s^{m+1}(-b_{22}\frac{m}{\tau} - b_{11}\frac{m}{\tau}) + s^m(a_{11}b_{22} +$$

$$+b_{22}\frac{m}{\tau} + b_{11}a_{22} + b_{11}\frac{m}{\tau} - a_{21}b_{12} - a_{12}b_{21}) +$$

$$+b_{11}b_{22} - b_{12}b_{21},$$

which is convenient for numerical root finding on the computer.

IV. On the Stability of Linear Systems with Delay

The study of the stability of solutions of linear systems with delay (5) is currently one of the most important for practical tasks.

Theorem 3[10]. *In order for the zero solution of system (8) to be exponentially stable, it is necessary and sufficient that all the roots of its quasipolynomial (9) lie in the left half-plane*

$$\operatorname{Re}\lambda < 0. \quad (13)$$

Direct calculation of the zeros of a quasipolynomial (6) or analysis of their localization is a rather difficult task, especially for high-order systems. The possibility of studying the stability (instability) of the system (8) by analyzing the approximating system of ordinary differential equations provides the following statement.

Theorem 4 [5]. *If zero solution of equation (1) is exponentially stable (not stable) then there exists $m_0>0$ such that for all $m>m_0$, zero solution of system (3) is exponentially stable (not stable). If for all $m>m_0$ zero solution of approximation system (3) is exponentially stable (not stable) then zero solution of equation (1) is exponentially stable (not stable).*

Remark 1. *From Theorem 4, the existing number is such that when the asymptotic stability (instability) of the zero solution of the system with delay (8)) is equivalent to the stability (instability) of the zero solution of the approximating system of ordinary differential equations (10).*

Theorem 4 can be applied for finding the upper limit of the delay in the system (8) which has stability. Calculating the approximate roots of the quasi-polynomial (9) with different τ, we can estimate the upper value of the delay τ, for which the system (8) is stable.

Using theorems 4 and lemma 1 we can obtain an effective algorithm for stability analysis of the system

$$\frac{dx}{dt} = Ax(t) + Bx(t-\tau), \quad (14)$$

where $x \in R^n$, $A, B - n\times n$ are fixed matrices, $\tau > 0$.

When evaluating zeros of the characteristic equation of the approximating system of ordinary differential equations for (14) with different values of τ remaining stability of zero solution of the approximating system, we find the delay domain τ, making system (14) to be exponentially stable.

In the case where system with delay (14) is of the second order, this algorithm is easy to apply.

Example2. Numerical experiments show us that system (14) with matrices

$$A = \begin{pmatrix} -0,9 & -6,5 \\ 4,8 & -0,9 \end{pmatrix}, \quad B = \begin{pmatrix} -1,39 & -0,65 \\ 0,48 & -1,39 \end{pmatrix}$$

is asymptotically stable if and only if $\tau \in (0,\tau_1)\cup(\tau_2,\tau_3)$, where $\tau_1 = 0,2862$, $\tau_2 = 0,7141$, $\tau_3 = 1,2142$.

V. Building Stability Region

Consider the linear differential-difference equation with delays

$$\frac{dx(t)}{dt} + \sum_{k=1}^{n} b_k x(t-\tau_k) = 0, \quad (15)$$

where $b_k \in \Box$, $k \in \{1,2,...,n\}$, $0 < \tau_1 < \tau_2 < ... < \tau_n = \tau$.

Let in the equation (15) delays τ_k are rational positive numbers. If we make linear substitution of independent value then it's possible to achieve another linear equation which corresponds to the characteristic equation of the form:

$$\lambda = a_1e^{-\lambda} + a_2e^{-2\lambda} + ... + a_ne^{-n\lambda}. \quad (16)$$

Theorem 5[11]. *Let in the equation (15) $b_k, \tau_k > 0, k = \overline{1,n}$ and inequality is true*

$$\sum_{k=1}^{n} b_k\tau_k < \frac{\pi}{2}.$$

Then all the roots of quasi-polynomial equation (15) have negative real parts.

The above Theorem 5 determines sufficient conditions for the stability of equation (15), but it does not allow to construct a coefficient region of stability, since the zero solution of equation (15) can be stable at an arbitrarily large value $\sum_{k=1}^{n} b_k\tau_k$ [11].

Definition [12]. *The stability region of equation (16) is called $(a_1,...,a_n) \in R_n$ the set of points for which all roots of equation (12) satisfy the condition $Re\lambda < 0$.*

Theorem 6 [12]. *The region of stability of equation (16) is bounded.*

Remark 2. *The region of stability of equation (15), where τ_k are positive rational numbers is bounded.*

Lemma 2[12]. If the vector $(a_1, a_2, ..., a_n)$ *belongs to the region of stability eguation (12), then* $a_1 + a_2, +... + a_n < 0$.

As an example, consider a linear differential equation with two delays

$$\frac{dx(t)}{dt} = a_1(t-m) + a_2(t-n), \qquad (17)$$

where $\{a_1, a_2\} \subset \mathbb{R}$, $m, n-$ are rational numbers.

From theorem 5 and lemma 2 we get that the stability region of equation (13) in the plane of the parameters a1, a2 is contained in a limited polygon

$$\| a_1 | - | a_2 \| < \pi, \ \| a_1 | e^{-m} - | a_2 | e^{-n} | < \sqrt{\pi^2 + 1}, \ a_1 + a_2 < 0.$$

We cover this area with a net of points $(a_{0i}, a_{1j}), i \in \{0,1,\ldots,p\}, j \in \{0,1,\ldots,k\}, \{n,k\} \subset \mathbb{N}$.

For every point (a_{0i}, a_{1j}) we have a linear equation with a delay of the form (17).

Example3. Consider a linear differential equation with two delays

$$\frac{dx(t)}{dt} = -a_0(t-0.4) - a_1(t-0.7),$$

where $\{a_0, a_1\} \subset \mathbb{R}$.

The results of numerical experiments at different steps of approximation grid h were obtained in Mathcad and are shown on figures 2 and 3. The region of stability is a shaded part of the plane.

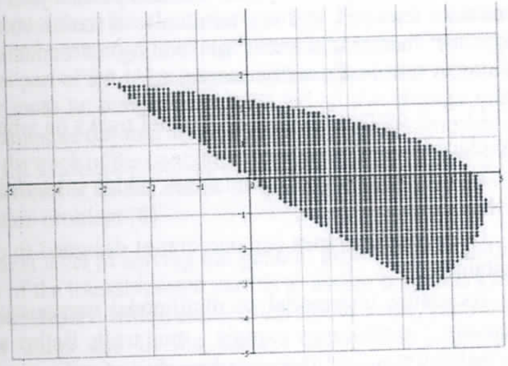

Fig. 2 (h=0.1)

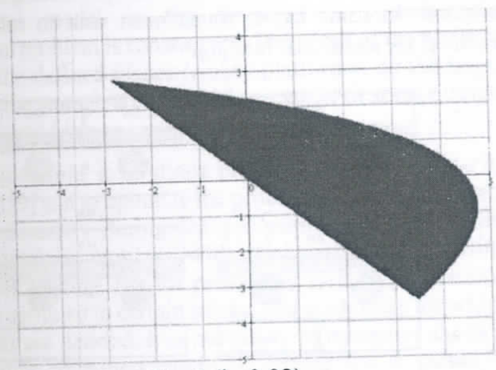

Fig. 3 (h=0.02)

VI. CONCLUSIONS

The work is devoted to analysis of approximation schemes regarding different classes of differential difference equations by means of special systems of ordinary differential equations. For the linear differential equations with delay, the approximation schemes allow to construct algorithms on approximate finding of non-asymptotic roots of the corresponding quasi-polynomials. Computational formulas convenient for computer application were obtained for scalar differential equations with one, two and three delays, as well as for second-order systems.

Calculating the roots of quasi-polynomials using the appropriate approximating polynomials can be performed with the built-in functions of Matlab, Maple, Mathematica or with the Numpy library on the Python platform.

Using the approximate finding algorithms for non-asymptotic roots of quasi-polynomials, a way for constructing the coefficient areas of stability for linear differential equations with delay and finding the set of delay values for which the equation is asymptotically stable is suggested.

Performed numerical experiments for model test examples confirm the effectiveness of proposed schemes for modeling the linear differential equations with delay.

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