Approximation schemes for differential functional equations and their applications

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In mathematical modeling of physical and technical processes, the evolution of which depends on prehistory, we arrive at differential equations with a delay. With the help of such equations it was possible to identify and describe new effects and phenomena in physics, biology, technology [1]. An important task for differential-functional equations is to construct and substantiate finding approximate solutions, since there are currently no universal methods for finding their precise solutions. Of particular interest are studies that allow the use of methods of the theory of ordinary differential equations for the analysis of delay differential equations.

Schemes for approximating differential-difference equations by special schemes of ordinary differential equations are proposed in the works [2,3]. Further research was found in I. M. Cherevko, L. A. Piddubna, O. V. Matwiy's works [4,5] in various functional spaces. Consider the Cauchy problem for a delayed differential equation

$$\frac{dx}{dt} = F(t, x(t), x(t-\tau)), \tag{1}$$

$$x(t) = \varphi(t), t \in [t_0 - \tau, t_0], \tag{2}$$

where $x \in R^n, \tau > 0, \tau_0 \in R, F(t, u, v)$ is a continuous function.

Equation (1) corresponds to an approximating system of ordinary differential equations

$$\frac{dt_0}{dt} = F(t, z_o, z_m), \tag{3}$$

$$\frac{dz_j}{dt} = \frac{m}{\tau}(t_{j-1}(t) - z_j(t)), j = \overline{1, m},$$

$$z_j(t_0) = \varphi(t_0 - \frac{j\tau}{m}), j = \overline{0, m}. \tag{4}$$

Теорема 1. *.[2,4] If the solution of problem (1)–(2) satisfies the Lipschitz condition on* $[t_0 - \tau, T]$*, then* $|x(t - \frac{j\tau}{m}) - z_j(t)| \le \frac{K\tau}{\sqrt{m}}, t_0 \in [t_0, T], K > 0.$

If the solution of the problem (1)-(2) $x(t) \in [t_0, T]$*, then* $|x(t - \frac{j\tau}{m}) - z_j(t)| \le \beta(\omega(x, \frac{\tau}{m})), j = \overline{0, m}, t \in [t_0, T]$*, where* $\beta(\delta) \to 0 at \delta \to 0, \omega(x, \frac{\tau}{m})$ *- the continuity modulus of the function* $x(t)$ *on* $[t_0 - \tau, T]$.

Note that according to Cantor's theorem on uniform continuity $\omega(x, \frac{\tau}{m}) \to 0$ when $m \to 0$. Therefore, for large the solution of the Cauchy problem for the system of ordinary differential equations (3) - (4) approximates the solution of the initial problem for the delay equation (1) - (2).

The study of approximation of linear stationary systems with a delay allowed us to construct algorithms for approximate detection of nonasymptotic roots of quasi-polynomials. Using these algorithms, a method for modelling the stability of solutions of linear systems with a delay is developed, as well as constructive computational algorithms for constructing coefficient regions of stability of linear systems with many delays [5-6].

Consider the initial problem for a linear system of differential-difference equations

$$\frac{dx}{dt} = Ax(t) + \sum_{i=1}^{k} B_i x(t - \tau_i), \tag{5}$$

$$x(t) = \varphi(t), t \in [-\tau, 0], \tag{6}$$

where $A, B_i, j = \overline{1,k}$ fixed $n \times n$ matrix $x \in R^n, 0 < \tau_1 < \tau_2 < ... < \tau_k = \tau, \varphi(t) \in [-\tau, 0]$.

Let us correspond to the initial problem (5) - (6) the system of ordinary differential equations

$$\frac{dz_0}{dt} = A(t)z_0(t) + \sum_{i=1}^{k} B_i z_{l_i}(t), l_i = \frac{\tau_i m}{\tau},$$

$$\frac{dz_j(t)}{dt} = \mu(z_j - 1(t) - z_j(t)), j = \overline{1,m}, \mu = \frac{m}{\tau}, m \in N, \tag{7}$$

with initial conditions

$$z_j(0) = \varphi(-\frac{\tau j}{m}), j = \overline{0,m}. \tag{8}$$

Теорема 2. *.[5] If the zero solution of the system with delay (1) is exponentially stable (not stable), then there is $m_0 > 0$ such that for all $m > m_0$, the zero solution of the approximating system (3) is also exponentially stable (not stable).*

If for all $m > m_0$ the zero solution of the approximation system (3) is exponentially stable (not stable) then the zero solution of the system with a delay (1) is exponentially stable (not stable).

It follows from Theorem 2 that the asymptotic stability or instability of the solutions of the delayed linear equations and the corresponding approximating system of ordinary differential equations for sufficiently large values of m are equivalent.

The obtained algorithms for finding non-asymptotic quasipolynomial roots and constructing regions of stability of linear delay differential equations can be used to study applied problems of optimal control, modeling of dynamic processes in economics, ecology and others [7].

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