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RESEARCH ARTICLE

Separately continuous functions with a given rectangular set of points of discontinuity

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Abstract
We investigate the problem of existence of a separately continuous function $f: X \times Y \to \mathbb{R}$ defined on a product of topological spaces X and Y with a given discontinuity points set of the form $A \times B$. Using an approach based on the classical Schwartz function we prove the existence of separately continuous function $f: X \times Y \to \mathbb{R}$ whose discontinuity points set is equal to the product $A \times B$ of nowhere dense functionally closed sets in a quite general case of spaces X and Y. Moreover, we show that if $X = Y = \beta\omega$, where $\beta\omega$ is the Čech–Stone compactification of the countable discrete space ω, and $A = B = \beta\omega \setminus \omega$, then there is no separately continuous function $f: X \times Y \to \mathbb{R}$ whose discontinuity points set is equal to $A \times B$.

Keywords Separately continuous function · Discontinuity points set · Čech–Stone compactification · Compact space

Mathematics Subject Classification 54C30 · 26B35 · 54C08 · 54D30

1 Introduction

Investigations of the discontinuity points set of separately continuous functions of two or many variables (i.e. functions that are continuous with respect to each variable) were started by René Baire [1]. He proved that the discontinuity points set $D(f)$ of

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any separately continuous function $f\colon [0,1]^2 \to \mathbb{R}$ is an F_σ-set which is contained in the product $A \times B$ of meager sets $A, B \subseteq [0,1]$. Richard Kershner [6] established a characterization of the discontinuity points set of separately continuous functions of real variables. He proved that the necessary conditions obtained by Baire were also sufficient conditions on the discontinuity points set of such functions.

Further development of investigations of the direct problem and the inverse problem (necessary and sufficient conditions on the discontinuity points set of separately continuous functions) was carried out in the following papers: Feiock [5], Breckenridge and Nishiura [2], Calbrix and Troallic [3]. This led to a similar characterization of the discontinuity points set of separately continuous functions defined on the product of separable metrizable spaces. To date, the most general results on a complete description of the discontinuity points set of separately continuous functions $f\colon X_1 \times \cdots \times X_n \to \mathbb{R}$ have been obtained in the following two cases only: if each space X_k is metrizable [7] and if each space X_k is the product of a family of separable metrizable spaces [8].

On the other hand, according to the Namioka Theorem [12], the discontinuity points set $D(f)$ of any separately continuous function f defined on the product of two compact spaces X and Y is contained in the product of meager sets. The question of characterization of the discontinuity points set of separately continuous functions defined on the product of two compact spaces arises naturally in connection with Namioka's result (this question was formulated by Piotrowski [13]). It follows from [9, Theorem 9] that Kershner's characterization theorem is not true for separately continuous functions defined on the product of two arbitrary compact spaces. Therefore it is important to study a *special inverse problem* of construction of a separately continuous function $f\colon X \times Y \to \mathbb{R}$ with a given discontinuity points set of special rectangular type $A \times B$, where $A \subseteq X$ and $B \subseteq Y$. This problem was investigated in [10, 11] where the following results were obtained.

Theorem 1.1 (a) *Let X be a topological space, Y be a locally connected space, A and B be nowhere dense functionally closed sets in X and Y respectively. Then there exists a separately continuous function $f\colon X \times Y \to \mathbb{R}$ whose discontinuity points set is equal to $A \times B$* [10].

(b) *Let X and Y be compact spaces, A and B be nowhere dense functionally closed sets in X and Y respectively. Then there exists a separately continuous function $f\colon X \times Y \to \mathbb{R}$ such that the projections on X and Y of the discontinuity points set of f are equal to A and B respectively* [11, Theorem 2.5].

(c) *There exist Eberlein compacts X and Y and nowhere dense functionally closed sets A and B in X and Y respectively such that $D(f) \neq A \times B$ for every separately continuous function $f\colon X \times Y \to \mathbb{R}$* [11, Theorem 3.2].

(d) (CH) *There exist separable Valdivia compacts X and Y and nowhere dense functionally closed sets A and B in X and Y respectively such that $D(f) \neq A \times B$ for every separately continuous function $f\colon X \times Y \to \mathbb{R}$* [11, Theorem 4.7].

This paper is devoted to investigation of the special inverse problem. First, we introduce and study the notion of a regular functionally closed set in a topological space. Using this notion and an approach based on the classical Schwartz function we prove the existence of separately continuous function $f\colon X \times Y \to \mathbb{R}$ whose discontinuity points set is equal to the product $A \times B$ of nowhere dense functionally closed sets in a quite general case of spaces X and Y. Moreover, we show that if $X = Y = \beta\omega$, where $\beta\omega$ is the Čech–Stone compactification of the countable discrete space ω, and $A = B = \beta\omega \setminus \omega$, then there is no separately continuous function $f\colon X \times Y \to \mathbb{R}$ whose discontinuity points set is equal to $A \times B$.

2 Regular sets in metrizable spaces

Recall that a set A in a topological space X is called *functionally closed*, if there exists a continuous function $\varphi\colon X \to [0, 1]$ such that $A = \varphi^{-1}(0)$.

A topological space X is called *locally connected* (see [4]), if for every $x \in X$ and any neighbourhood U of x there exists a connected neighbourhood $V \subseteq U$ of x.

We say that a subset A in a topological space X is *regular*, if there exists a continuous function $\varphi\colon X \to [0, 1]$ such that

(a) $A = \varphi^{-1}(0)$;
(b) for every point $a \in A$ and a neighborhood U of a there exists an integer $n \in \mathbb{N}$ such that

$$\varphi(U) \cap \left[\frac{1}{2^k}, \frac{1}{2^{k-1}}\right] \neq \varnothing$$

for every $k \geqslant n$.

It is clear that every regular set in a topological space X is a nowhere dense functionally closed set in X. On the other hand, the following proposition is true.

Proposition 2.1 *Any nowhere dense functionally closed set A in a locally connected space X is a regular set in X.*

Proof Let $\varphi\colon X \to [0, 1]$ be a continuous function such that $A = \varphi^{-1}(0)$. We fix a point $a \in A$ and a neighborhood U of a in X.

Choose a connected neighborhood $V \subseteq U$ of a in X. Since A is nowhere dense, $V \setminus A \neq \varnothing$ and $\varphi(V) \neq \{0\}$. Moreover, the set $\varphi(V)$ is a connected subset of $[0, 1]$ as a continuous image of a connected set. Therefore, there exists an $\varepsilon > 0$ such that $[0, \varepsilon) \subseteq \varphi(V)$. Then

$$\varphi(V) \cap \left[\frac{1}{2^k}, \frac{1}{2^{k-1}}\right] \neq \varnothing$$

for every $k \geqslant \left[\log_2 \frac{1}{\varepsilon}\right]$. □

Proposition 2.2 *The union of finite family of pairwise disjoint regular subsets of a topological space is a regular set.*

Proof We consider the case of the union of two disjoint regular sets A and B in a topological space X only.

Let continuous functions $\varphi: X \to [0,1]$ and $\psi: X \to [0,1]$ satisfy conditions (a) and (b) from the definition of a regular set for sets A and B respectively. It is enough to consider the continuous function

$$\chi(x) = \min\{\varphi(x), \psi(x)\}.$$

□

Question 2.3 Let A and B be regular subsets of a topological space X. Is it true that $A \cap B$ and $A \cup B$ are regular?

We need the following fact which has a simple proof.

Lemma 2.4 *Let $(t_n)_{n=1}^{\infty}$ be a strictly decreasing sequence of real numbers $t_n \in (0,1]$ such that*

$$\lim_{n\to\infty} t_n = 0.$$

Then there exists a strictly increasing continuous function $\psi: [0,1] \to [0,1]$ such that $\psi(0) = 0$ and $\psi(t_n) = 1/2^n$ for every $n \in \mathbb{N}$.

Proposition 2.5 *Suppose that a point $a \in X$ has a countable base of neighborhoods in a topological space X and the set $A = \{a\}$ is a nowhere dense functionally closed set in X. Then A is regular.*

Proof Let $\varphi: X \to [0,1]$ be a continuous function such that

$$A = \varphi^{-1}(0).$$

Since A is nowhere dense and a has a countable base of neighborhoods, there exists a sequence $(x_n)_{n=1}^{\infty}$ of points $x_n \in X$ such that $x_n \to a$ in X and the sequence $(\varphi(x_n))_{n=1}^{\infty}$ converges to zero while strictly decreasing. For every $n \in \mathbb{N}$ we put $t_n = \varphi(x_n)$. According to Lemma 2.4, we choose a strictly increasing continuous function $g: [0,1] \to [0,1]$ such that $g(0) = 0$ and $g(t_n) = 1/2^n$ for every $n \in \mathbb{N}$. It remains to consider a continuous function $\varphi: X \to [0,1]$,

$$\psi(x) = g(\varphi(x)).$$

□

Let us get down to studying regular sets in metrizable spaces.

In a metric space (X, d) by $B(x, r)$ we denote an open ball with the center $x \in X$ and radius $r > 0$.

We start from the following auxiliary statements.

Lemma 2.6 *Let (X, d) be a metric space, $\delta > 0$, $A \subseteq X$ be a closed nowhere dense set and $G \subseteq X$ be an open set such that $A \subseteq G$. Then there exists an open set $H \subseteq X$ such that*

$$A \subseteq H \subseteq \overline{H} \subseteq G$$

and

$$B(a,\delta)\cap(G\setminus\overline{H})\neq\varnothing$$

for every $a\in A$.

Proof For every $a\in A$ let us choose a real number $\delta_a<\delta/2$ such that $B(a,\delta_a)\subseteq G$. According to the Stone Theorem [4, Theorem 4.4.1], there exists a locally finite in X open cover $\mathfrak{U}$ of A which is inscribed in the open cover $\{B(a,\delta_a):a\in A\}$ of A. For every $U\in\mathfrak{U}$ we choose a point $b_U\in U\setminus A$ and consider the set

$$B=\{b_U:U\in\mathfrak{U}\}.$$

Since the system $\mathfrak{U}$ is locally finite, the set B is closed. Moreover, $B\subseteq G\setminus A$, in particular, $B\cap A=\varnothing$. Since X is a normal space, there exists an open set H such that

$$A\subseteq H\subseteq\overline{H}\subseteq G\setminus B.$$

It remains to show that $B(a,\delta)\cap(G\setminus\overline{H})\neq\varnothing$ for every $a\in A$. Fix a point $a\in A$ and choose $U\in\mathfrak{U}$ such that $a\in U$. Then there exists $a_1\in A$ such that $U\subseteq B(a_1,\delta_{a_1})$. Then we have

$$d(a,b_U)\leqslant d(a,a_1)+d(a_1,b_U)\leqslant\delta_{a_1}+\delta_{a_1}\leqslant\frac{\delta}{2}+\frac{\delta}{2}=\delta.$$

Thus, $b_U\in B\cap B(a,\delta)$. □

Lemma 2.7 *Let* $(G_n)_{n=1}^{\infty}$ *be a sequence of open nonempty sets* G_n *in a normal space* X *such that*

$$\overline{G}_{n+1}\subseteq G_n$$

for every $n\in\mathbb{N}$. *Then there exists a continuous function* $\varphi\colon X\to[0,1)$ *such that*

$$\frac{1}{2^n}\leqslant\varphi(x)\leqslant\frac{1}{2^{n-1}}$$

for every $n\in\mathbb{N}$ *and* $x\in\overline{G}_n\setminus G_{n+1}$, *and*

$$A=\bigcap_{n=1}^{\infty}G_n=\varphi^{-1}(0).$$

Proof According to the Urysohn Lemma [4, Theorem 1.5.11], for every $n\in\mathbb{N}$ there exists a continuous function $\varphi_n:X\to[0,1/2^n]$ such that

$$\overline{G}_{n+1}\subseteq\varphi_n^{-1}(0)\quad\text{and}\quad X\setminus G_n\subseteq\varphi_n^{-1}\left(\frac{1}{2^n}\right).$$

We consider the function $\varphi\colon X \to [0, 1]$,

$$\varphi(x) = \sum_{n=1}^{\infty} \varphi_n(x).$$

Clearly, φ is a continuous function. Let $n \in \mathbb{N}$ and $x \in \overline{G}_n \setminus G_{n+1}$. Then $\varphi_k(x) = 0$ for every $k \in \{1, \dots, n-1\}$ and $\varphi_k(x) = 1/2^k$ for every $k \geqslant n+1$. Thus, we have

$$\varphi(x) = \sum_{k=1}^{\infty} \varphi_k(x) = \varphi_n(x) + \sum_{k=n+1}^{\infty} \frac{1}{2^k} = \varphi_n(x) + \frac{1}{2^n}.$$

Since $\varphi_n(x) \in [0, 1/2^n]$,

$$\frac{1}{2^n} \leqslant \varphi(x) \leqslant \frac{1}{2^{n-1}}.$$

□

The following proposition is the main proposition of this section.

Proposition 2.8 *Every closed nowhere dense set A in a metrizable space X is regular.*

Proof Let us fix a metric d on X which generates the topology of X. It follows from Lemma 2.6 that there exists a sequence $(G_n)_{n=1}^{\infty}$ of open sets G_n in X which satisfies the following conditions:

(α) $A \subseteq G_{n+1} \subseteq \overline{G}_{n+1} \subseteq G_n$ for every $n \in \mathbb{N}$;
(β) $B(a, 1/n) \cap (G_n \setminus \overline{G}_{n+1}) \neq \varnothing$ for every $n \in \mathbb{N}$ and $a \in A$.

According to Lemma 2.7, there exists a continuous function $\varphi\colon X \to [0, 1]$ such that $A = \varphi^{-1}(0)$ and

$$\frac{1}{2^n} \leqslant \varphi(x) \leqslant \frac{1}{2^{n-1}}$$

for every $n \in \mathbb{N}$ and $x \in \overline{G}_n \setminus G_{n+1}$.

Let us verify condition (b) from the definition of a regular set. Fix an element $a \in A$ and a real number $\delta > 0$ and choose an integer m such that $1/m \leqslant \delta$. Then

$$B\left(a, \frac{1}{k}\right) \subseteq B\left(a, \frac{1}{m}\right) \subseteq B(a, \delta) = U$$

for every $k \geqslant m$. According to (β), we have

$$B\left(a, \frac{1}{k}\right) \cap (G_k \setminus \overline{G}_{n+1}) \neq \varnothing.$$

It follows from the properties of the function φ that

$$\varphi(G_k \setminus \overline{G}_{n+1}) \subseteq \left[\frac{1}{2^k}, \frac{1}{2^{k-1}}\right].$$

Thus,

$$\varphi(U) \cap \left[\frac{1}{2^k}, \frac{1}{2^{k-1}}\right] \neq \varnothing$$

for every $k \geqslant n$. □

3 Separable regular sets

We start with the following auxiliary statement.

Lemma 3.1 *Let $(t_{nk} : n \in \mathbb{N}, k \in \mathbb{N})$ be a family of real numbers $t_{nk} \in (0, 1]$ such that $\lim_{k\to\infty} t_{nk} = 0$ for every $n \in \mathbb{N}$. Then there exists a strictly decreasing sequence $(t_n)_{n=1}^{\infty}$ of real numbers $t_n \in (0, 1]$ such that*

(α) $\lim_{n\to\infty} t_n = 0$*;*
(β) *for all $n \in \mathbb{N}$ and $m \leqslant n$ there exists $k \in \mathbb{N}$ such that*

$$t_{n+1} < t_{mk} \leqslant t_n.$$

Proof Let us construct the sequence $(t_n)_{n=1}^{\infty}$ inductively. First, we put $t_1 = 1$ and choose $t_2 \in (0, t_1/2]$ such that

$$\{t_{1k} : k \in \mathbb{N}\} \cap (t_2, t_1] \neq \varnothing.$$

Further, since $t_{nk} > 0$ and $\lim_{k\to\infty} t_{1k} = \lim_{k\to\infty} t_{2k} = 0$, there exists $t_3 \in (0, t_2/2]$ such that

$$\{t_{mk} : k \in \mathbb{N}\} \cap (t_3, t_2] \neq \varnothing$$

for every $m = 1, 2$. And so on.

It is clear that the sequence $(t_n)_{n=1}^{\infty}$ satisfies condition (β). Moreover, $t_{n+1} \leqslant t_1/2^n$ for every $n \in \mathbb{N}$. Therefore, condition (α) is also true. □

We need the following notion. We say that a set A in a topological space X is *bilaterally separable* if there exist a sequence $(a_n)_{n=1}^{\infty}$ of points $a_n \in A$ and a family $(x_{nk} : n \in \mathbb{N}, k \in \mathbb{N})$ of points $x_{nk} \in X \setminus A$ such that

- $A \subseteq \overline{\{a_n : n \in \mathbb{N}\}}$;
- $\lim_{k\to\infty} x_{nk} = a_n$ for every $k \in \mathbb{N}$.

Clearly, every nowhere dense separable set in a first countable topological space is bilaterally separable.

The following property of bilaterally separable functionally closed sets plays an important role in our further construction.

Proposition 3.2 *Let X be a topological space and $A \subseteq X$ be a functionally closed bilaterally separable set. Then A is regular.*

Proof Let $\varphi_1 : X \to [0,1]$ be a continuous function such that $A = \varphi_1^{-1}(0)$, $(a_n)_{n=1}^{\infty}$ be a sequence of points $a_n \in A$ and $(x_{nk} : n \in \mathbb{N}, k \in \mathbb{N})$ be a family of points $x_{nk} \in X \setminus A$ such that $A \subseteq \overline{\{a_n : n \in \mathbb{N}\}}$ and $\lim_{k\to\infty} x_{nk} = a_n$ for every $k \in \mathbb{N}$.

For all $n, k \in \mathbb{N}$ we put

$$t_{nk} = \varphi_1(x_{nk}).$$

Notice that

$$\lim_{k\to\infty} t_{nk} = \lim_{k\to\infty} \varphi_1(x_{nk}) = \varphi_1(a_n) = 0$$

for every $n \in \mathbb{N}$, that is the family $(t_{nk} : n \in \mathbb{N}, k \in \mathbb{N})$ satisfies the conditions of Lemma 3.1. Therefore, there exists a strictly decreasing null sequence $(t_n)_{n=1}^{\infty}$ of real numbers $t_n \in (0,1]$ such that for all $n \in \mathbb{N}$ and $m \leqslant n$ there exists $k \in \mathbb{N}$ with

$$t_{n+1} < t_{mk} \leqslant t_n.$$

According to Lemma 2.4, there exists a strictly increasing continuous function $\psi : [0,1] \to [0,1]$ such that $\psi(0) = 0$ and $\psi(t_n) = 1/2^n$ for every $n \in \mathbb{N}$.

Consider a continuous function $\varphi : X \to [0,1]$

$$\varphi(x) = \psi(\varphi_1(x)).$$

Notice that

$$\varphi^{-1}(0) = \varphi_1^{-1}(\psi^{-1}(0)) = \varphi_1^{-1}(0) = A.$$

Let us verify condition (b) from the definition of regular set. Choose an arbitrary point $a \in A$ and an open neighborhood U of a. Since the set $\{a_n : n \in \mathbb{N}\}$ is dense in A, there is an integer $n_1 \in \mathbb{N}$ such that $a_{n_1} \in U$. Moreover,

$$\lim_{k\to\infty} x_{n_1,k} = a_{n_1}.$$

Therefore, there is an integer $k_0 \in \mathbb{N}$ such that $x_{n_1,k} \in U$ for every $k \geqslant k_0$. According to the choice of the sequence $(t_n)_{n=1}^{\infty}$, for every $n \geqslant n_1$ there exists $k_n \in \mathbb{N}$ such that

$$t_{n+1} < t_{n_1,k_n} \leqslant t_n.$$

It is clear that all integers k_n are distinct. Therefore, there exists $n_0 \geqslant n_1$ such that $k_n \geqslant k_0$ for all $n \geqslant n_0$. Then $x_{n_1,k_n} \in U$ for every $n \geqslant n_0$ and

$$\frac{1}{2^{n+1}} = \psi(t_{n+1}) < \psi(t_{n_1,k_n}) = \varphi(x_{n_1,k_n}) \leqslant \psi(t_n) = \frac{1}{2^n},$$

that is,

$$x_{n_1,k_n} \in \psi(U) \cap \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right]. \qquad \square$$

4 Construction of separately continuous functions with a given rectangular set of points of discontinuity

Let X be a topological space, $A \subseteq X$ be a nonempty set, $x_0 \in X$, $\mathfrak{U}$ be the system of all neighborhoods of x_0 in X and $f: X \to \mathbb{R}$. The real number

$$\omega_f(A) = \sup_{x',x'' \in A} |f(x') - f(x'')|$$

is called the *oscillation of the function f on the set A*, and the real number

$$\omega_f(x_0) = \inf_{U \in \mathfrak{U}} \omega_f(U)$$

is called the *oscillation of the function f at the point x_0*.

For a mapping f between topological spaces the discontinuity points set of f we denote by $D(f)$.

The following theorem gives a solution to the special inverse problem for the product $A \times B$ where A is regular.

Theorem 4.1 *Let X, Y be topological spaces, $A \subseteq X$ be a regular set in X and $B \subseteq Y$ be a nowhere dense functionally closed set in Y. Then there exists a lower semi-continuous separately continuous function $f: X \times Y \to [0, 1]$ such that $D(f) = A \times B$.*

Proof Let $\varphi: X \to [0, 1]$ be a continuous function which satisfies conditions (a) and (b) from the definition of regular set. Moreover, let $\psi: Y \to [0, 1]$ be a continuous function such that $B = \psi^{-1}(0)$. Consider a continuous function

$$f(x, y) = \begin{cases} \frac{2\varphi(x)\psi(y)}{\varphi^2(x)+\psi^2(y)}, & (x, y) \notin A \times B, \\ 0, & (x, y) \in A \times B. \end{cases}$$

Since the Schwartz function

$$g(s, t) = \begin{cases} \frac{2st}{s^2+t^2}, & (s, t) \neq (0, 0), \\ 0, & (s, t) = (0, 0), \end{cases}$$

is continuous at every point $(s, t) \neq (0, 0)$, the function f is continuous at every point $(x, y) \notin A \times B$. Moreover,

$$\varphi(x, y) = 0 = \min\{f(z) : z \in X \times Y\}$$

for every $(x, y) \in A \times B$. Consequently, f is lower semi-continuous at every point $(x, y) \in A \times B$. Therefore, f is lower semi-continuous.

It remains to show that f is discontinuous at every point $(x, y) \in A \times B$. Fix any point $(x_0, y_0) \in A \times B$ and arbitrary open neighborhoods U of x_0 in X and V of y_0 in Y. According to condition (b), choose an integer $n \in \mathbb{N}$ such that

$$\varphi(U) \cap \left[\frac{1}{2^k}, \frac{1}{2^{k-1}}\right] \neq \varnothing$$

for all $k \geqslant n$. Consider the open neighborhood

$$V_0 = V \cap \psi^{-1}\left(\left[0, \frac{1}{2^{n-1}}\right)\right)$$

of y_0 in Y. Since the set B is nowhere dense, $V_0 \setminus B \neq \varnothing$. Thus, there exists a point $y_1 \in V_0 \setminus B$. Clearly, $\psi(y_1) \in (0, 1/2^{n-1})$. Choose $k \geqslant n$ such that

$$\frac{1}{2^k} \leqslant \psi(y_1) \leqslant \frac{1}{2^{k-1}}.$$

According to the choice of n, there exists $x_1 \in U$ such that

$$\frac{1}{2^k} \leqslant \varphi(x_1) \leqslant \frac{1}{2^{k-1}}.$$

Then

$$f(x_1, y_1) = \frac{2\varphi(x_1)\psi(y_1)}{\varphi^2(x_1) + \psi^2(y_1)} \geqslant \frac{2\,\frac{1}{2^k}\,\frac{1}{2^k}}{\left(\frac{1}{2^{k-1}}\right)^2 + \left(\frac{1}{2^{k-1}}\right)^2} = \frac{1}{4}.$$

Therefore,

$$\omega_f(U \times V) \geqslant f(x_1, y_1) - f(x_0, y_0) \geqslant \frac{1}{4}$$

and

$$\omega_f(x_0, y_0) \geqslant \frac{1}{4}.$$

□

The following result follows immediately from Theorem 4.1, Propositions 2.8 and 5.4.

Corollary 4.2 *Let X, Y be topological spaces, $A \subseteq X, B \subseteq Y$ be nowhere dense functionally closed sets and one of the following conditions holds:*

(i) *X is locally connected;*
(ii) *X is metrizable;*

(iii) *A is bilaterally separable in X;*
(iv) *X is a first countable space and A is separable.*

Then there exists a lower semi-continuous separately continuous function $f\colon X\times Y\to[0,1]$ with $D(f)=A\times B$.

Remark 4.3 It follows from [11, Theorem 2.1] the existence of a lower semi-continuous separately continuous function $f\colon X\times Y\to[0,1]$ with $D(f)=A\times B$ in the case of nowhere dense functionally closed sets A and B in completely regular spaces X and Y such that $A\times B=\overline{\{p_n : n\in\mathbb{N}\}}$ and that every point p_n has a countable base (more generally, has the weak Preiss–Simon property) in $X\times Y$. But [11, Theorem 2.1] does not imply Corollary 4.2(iv) because we cannot use this theorem in the case of non-separable set B.

5 The special inverse problem on $\beta\omega\times\beta\omega$

In this section we consider the special inverse problem on the product $\beta\omega\times\beta\omega$, where $\beta\omega$ is the Čech–Stone compactification of the countable discrete space

$$\omega=\{0,1,2,\dots\}.$$

Notice that the set $\omega^*=\beta\omega\setminus\omega$ is a nowhere dense functionally closed set in a separable space $\beta\omega$.

Proposition 5.1 *The set ω^* is not regular in the space $\beta\omega$.*

Proof Let $\varphi\colon\beta\omega\to[0,1]$ be a continuous function such that $\omega^*=\varphi^{-1}(0)$. We set

$$A=\varphi^{-1}\left(\bigcup_{n=1}^{\infty}\left(\frac{1}{2^{4n-2}},\frac{1}{2^{4n-4}}\right]\right),\quad B=\varphi^{-1}\left(\bigcup_{n=1}^{\infty}\left(\frac{1}{2^{4n}},\frac{1}{2^{4n-2}}\right]\right),$$

$U=\overline{A}$ and $V=\overline{B}$. Clearly, $\omega=A\sqcup B$. According to [4, Corollary 3.6.4], $\beta\omega=U\sqcup V$. Therefore, U and V are open in $\beta\omega$. Moreover,

$$\varphi(U)\cap\left[\frac{1}{2^{4n-1}},\frac{1}{2^{4n-2}}\right]=\varnothing\quad\text{and}\quad\varphi(V)\cap\left[\frac{1}{2^{4n-3}},\frac{1}{2^{4n-4}}\right]=\varnothing$$

for every $n\in\mathbb{N}$. Thus, ω^* is not regular in $\beta\omega$. □

The question of existence of a separately continuous function $f\colon\beta\omega\times\beta\omega\to\mathbb{R}$ with $D(f)=\omega^*\times\omega^*$ naturally arises in connection with Theorem 4.1 and Proposition 5.1. We show that the answer to this question is negative.

We need the following auxiliary statements which have trivial proofs.

Lemma 5.2 *Let X be a topological space, $f\colon X\to\mathbb{R}$, $G\subseteq X$ be an open nonempty set, $g=f|_G$ and $x\in G$. Then $\omega_g(x)=\omega_f(x)$.*

Lemma 5.3 *Let X, Y be topological spaces, $\varphi\colon Y \to X$ be homeomorphism, $f\colon X \to \mathbb{R}$ and $g\colon Y \to \mathbb{R}$, $g(y) = f(\varphi(y))$. Then $\omega_g(y) = \omega_f(\varphi(y))$ for all $y \in Y$.*

For any nonempty set $A \subseteq \omega$ the closure $U = \overline{A}$ of the set A in the space $\beta\omega$ is called a *basic set in $\beta\omega$*. It is well-known [4, Theorem 3.6.13] that all basic sets are clopen in $\beta\omega$ and form a base of the topology of the space $\beta\omega$.

The following proposition give us a possibility to consider mappings with "big" oscillations and almost zero values on $\omega^* \times \omega^*$.

Proposition 5.4 *The following conditions are equivalent:*

(a) *there exists a separately continuous function $f\colon \beta\omega \times \beta\omega \to [0, 1]$ with $D(f) = \omega^* \times \omega^*$;*
(b) *there exists a separately continuous function $g\colon \beta\omega \times \beta\omega \to [0, 1]$ such that $D(g) = \omega^* \times \omega^*$, $g(\omega^* \times \omega^*) \subseteq (0, 1/4]$ and $\omega_g(x, y) \geqslant 1/2$ for every point $(x, y) \in \omega^* \times \omega^*$.*

Proof It is clear that (b) $\Rightarrow$ (a). Therefore, it remains to prove the implication (a) $\Rightarrow$ (b) only.

Let $f\colon \beta\omega \times \beta\omega \to [0, 1]$ be a separately continuous function with $D(f) = \omega^* \times \omega^*$. For every $n \in \mathbb{N}$ we put

$$F_n = \left\{(x, y) \in (\beta\omega)^2 : \omega_f(x, y) \geqslant \frac{1}{n}\right\}.$$

Since all sets F_n are closed, the space $\omega^* \times \omega^*$ is Baire and

$$\bigcup_{n=1}^{\infty} F_n = D(f) = \omega^* \times \omega^*,$$

there exist an integer $n_0 \in \mathbb{N}$ and infinite basic clopen sets U_0 and V_0 in $\beta\omega$ such that

$$\omega_f(x, y) \geqslant \frac{1}{n_0}$$

for every $(x, y) \in (\omega^* \cap U_0) \times (\omega^* \cap V_0)$. According to [4, Corollary 3.6.8], there exist homeomorphisms $\varphi\colon U_0 \to \beta\omega$ and $\psi\colon V_0 \to \beta\omega$, in particular $\varphi(U_0 \cap \omega^*) = \omega^*$ and $\psi(V_0 \cap \omega^*) = \omega^*$.

We consider the mapping $f_1\colon \beta\omega \times \beta\omega \to [0, 1]$,

$$f_1(x, y) = f(\varphi^{-1}(x), \psi^{-1}(y)).$$

It follows from Lemmas 5.2 and 5.3 that f_1 is separately continuous, $D(f_1) = \omega^* \times \omega^*$ and $\omega_{f_1}(x, y) \geqslant 1/n_0$ for every $(x, y) \in \omega^* \times \omega^*$.

Now we consider the restriction $h = f_1|_{\omega^* \times \omega^*}$ of f_1 on the product $\omega^* \times \omega^*$. It is clear that h is a separately continuous function which has a continuity point according to Namioka's Theorem [12]. Therefore, there exist infinite clopen basic sets $U_1 \subseteq \beta\omega$ and $V_1 \subseteq \beta\omega$ such that

$$\omega_h((U_1 \cap \omega^*) \times (V_1 \cap \omega^*)) \leqslant \frac{1}{2n_0}.$$

For a separately continuous mapping $f_2 : \beta\omega \times \beta\omega \to [0, 1]$,

$$f_2(x, y) = f_1(\varphi_1^{-1}(x), \psi_1^{-1}(y)),$$

where $\varphi_1 : U_1 \to \beta\omega$ and $\psi_1 : V_1 \to \beta\omega$ are some homeomorphisms, we have

$$D(f_2) = \omega^* \times \omega^*, \quad \omega_{f_2}(x, y) \geqslant \frac{1}{n_0}$$

for every $(x, y) \in \omega^* \times \omega^*$ and

$$f_2(\omega^* \times \omega^*) \subseteq \left[\alpha, \alpha + \frac{1}{2n_0}\right],$$

where $\alpha = \inf h((U_1 \cap \omega^*) \times (V_1 \cap \omega^*))$.

It remains to put

$$g(x, y) = \min\left\{1, \frac{n_0}{2}(f_2(x, y) - \alpha)\right\}$$

for every $(x, y) \in \beta\omega \times \beta\omega$. □

Let us get down to studying separately continuous functions that have almost zero values on $\omega^* \times \omega^*$.

Proposition 5.5 *Let $A, B \subseteq \omega$ be infinite sets and $f : \beta\omega \times \beta\omega \to [0, 1]$ be a separately continuous function such that*

$$f(\omega^* \times \omega^*) \subseteq \left[0, \frac{1}{4}\right].$$

Then there exists $n \in A$ such that the set

$$B_n = \left\{m \in B : f(n, m) \leqslant \frac{1}{3}\right\}$$

is infinite.

Proof Assume that for every $n \in A$ the set B_n is finite. Pick any point $y_0 \in \overline{B} \setminus B \subseteq \omega^*$. Then

$$\overline{B \setminus B_n} = \overline{B} \setminus B_n \supseteq \overline{B} \setminus B \ni y_0$$

for every $n \in A$. Since for every $n \in A$ we have

$$B \setminus B_n = \left\{ m \in B : f(n,m) > \frac{1}{3} \right\}$$

and the function $f^n : \beta\omega \to [0,1]$, $f^n(m) = f(n,m)$, is continuous,

$$f(n, y_0) = f^n(y_0) \in f^n(\overline{B \setminus B_n}) \subseteq \overline{f^n(B \setminus B_n)} \subseteq \left[\frac{1}{3}, 1\right].$$

Thus,

$$f(n, y_0) \geqslant \frac{1}{3}$$

for every $n \in A$. Taking into account the continuity of the function $f_{y_0} : \beta\omega \to [0,1]$, $f_{y_0}(x) = f(x, y_0)$, we obtain

$$f(x_0, y_0) \geqslant \frac{1}{3}$$

for every $x_0 \in \overline{A} \setminus A \subseteq \omega^*$, a contradiction. □

Proposition 5.6 *Let $f : \beta\omega \times \beta\omega \to [0,1]$ be a separately continuous function with $f(\omega^* \times \omega^*) \subseteq [0, 1/4]$. Then there exist infinite sets $A, B \subseteq \omega$ such that $f(A \times B) \subseteq [0, 1/3]$.*

Proof We put $A_1 = B_1 = \omega$. According to Proposition 5.5 there exists $n_1 \in A_1$ such that the set

$$B_2 = \left\{ m \in B_1 : f(n_1, m) \leqslant \frac{1}{3} \right\}$$

is infinite. Using Proposition 5.5 for sets $\widetilde{A}_1 = A_1 \cap (n_1, +\infty)$ and B_2 we obtain that there exists an integer $m_1 \in B_2$ such that the set

$$A_2 = \left\{ n \in \widetilde{A}_1 : f(n, m_1) \leqslant \frac{1}{3} \right\}$$

is infinite. Now we use Proposition 5.5 for sets A_2 and $\widetilde{B}_2 = B_2 \cap (m_1, +\infty)$ and obtain the existence of an integer $n_2 \in A_2$ such that the set

$$B_3 = \left\{ m \in \widetilde{B}_2 : f(n_2, m) \leqslant \frac{1}{3} \right\}$$

is infinite. And so on.

As a result, we obtain strictly increasing sequences $(n_k)_{k=1}^{\infty}$ and $(m_k)_{k=1}^{\infty}$ such that

$$f(n_k, m_l) \geqslant \frac{1}{3}$$

for every $k, l \in \omega$. It remains to put

$$A = \{n_k : k \in \omega\} \quad \text{and} \quad B = \{m_k : k \in \omega\}. \qquad \square$$

The following theorem is the main result of this section.

Theorem 5.7 *There is no separately continuous function* $f : \beta\omega \times \beta\omega \to [0, 1]$ *with* $D(f) = \omega^* \times \omega^*$.

Proof Assume that there exists a separately continuous function $f : \beta\omega \times \beta\omega \to [0, 1]$ with $D(f) = \omega^* \times \omega^*$. Then according to Proposition 5.4 there exists a separately continuous function $g : \beta\omega \times \beta\omega \to [0, 1]$ such that $g(\omega^* \times \omega^*) \subseteq [0, 1/4]$, $D(g) = \omega^* \times \omega^*$ and $\omega_g(x, y) \geqslant 1/2$ for every $(x, y) \in \omega^* \times \omega^*$.

It follows from Proposition 5.6 that there exist infinite sets $A, B \subseteq \omega$ such that $g(A \times B) \subseteq [0, 1/3]$. Since g is separately continuous,

$$g(\overline{A} \times \overline{B}) \subseteq \left[0, \frac{1}{3}\right].$$

Now for any points $x_0 \in \overline{A} \setminus A \subseteq \omega^*$ and $y_0 \in \overline{B} \setminus B \subseteq \omega^*$ the set $\overline{A} \times \overline{B}$ is a neighborhood of (x_0, y_0) in $\beta\omega \times \beta\omega$ and

$$\omega_g(x_0, y_0) \leqslant \omega_g(\overline{A} \times \overline{B}) \leqslant \frac{1}{3},$$

a contradiction. $\square$

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