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Research Article

The Cauchy Problem for Parabolic Equations with Degeneration

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Annotation. For a second-order parabolic equation, the multipoint in time Cauchy problem is considered. The coefficients of the equation and the boundary condition have power singularities of arbitrary order in time and space variables on a certain set of points. Conditions for the existence and uniqueness of the solution of the problem in Hölder spaces with power weight are found.

1. Introduction

One of the important directions of development of the modern theory of partial differential equations is the study of nonlocal boundary value problems for different types of differential equations and partial differential equation systems and establishing the conditions for their correct solvability. Such problems arise when modeling different phenomena and processes in modern science, quantum mechanics, technology, economics, etc.

Multipoint problems for partial differential equations were studied in the papers by Ptashnyk and his disciples. In particular, paper [1] is dedicated to multipoint problems for partial differential equations, not resolved relative to the highest time derivative. The question of the existence and qualitative properties of solutions for equations with limited order of degeneration has been studied in papers [2–4]. The Dirichlet problem with impulse action for a parabolic equation with power singularities of arbitrary order on time and spatial variables is investigated in paper [5]. In paper [6], a multipoint one-sided boundary value problem is studied. In paper [7], presented is the result of research of optimal control of the system described by the problem with an oblique derivative and an integral condition on time variable for parabolic equations with power singularities of arbitrary order. For a second-order parabolic equation, a multipoint (in time) problem with oblique derivative is considered in paper [8]. Conditions for the existence and uniqueness of solution of the posed problem in Hölder spaces with power weight are established.

In this paper, we investigate a multipoint time-varying Cauchy problem for a parabolic second-order equation with power singularities and arbitrary order degenerations in the coefficients of spatial and time variables at some set of points. We also find conditions for the existence and uniqueness of the solution of formulated problem in Hölder spaces with power weight.

2. Statement of the Problem and Main Result

Let $\eta, t_0, t_1, \cdots, t_N, t_{N+1}$ be fixed positive numbers, $0 \le t_0 < \cdots < t_{N+1}$, $t_0 < \eta < t_{N+1}$, $\eta \ne t_\lambda$, $\lambda \in \{1, 2, \cdots, N\}$, and let Ω be some bounded domain in R^{n-1}, $Q_{(0)} = \{(t, x) \mid t \in [t_0 ; t_{N+1}), x \in \Omega\} \cup \{(t, x) \mid t = \eta, x \in R^n \setminus \bar{\Omega}\}$.

Let us consider the problem of finding a function $u(x, t)$ in the domain $\Pi = [t_0, t_{N+1}) \times R^n$, which, for $t \ne t_\lambda$, $(t, x) \in Q_{(0)}$ satisfies the equation

$$\left[\partial_t - \sum_{i,j=1}^{n} A_{ij}(t, x)\partial_{x_i}\partial_{x_j} + \sum_{i=1}^{n} A_i(t, x)\partial_{x_i} + A_0(t, x)\right] u(t, x) = f(t, x), \tag{1}$$

and multipoint conditions for variable t

$$u(t_k+0,x)=\varphi_k(x), \quad k\in\{0,1,\cdots,N\}. \tag{2}$$

The power singularities of coefficients of the differential equation (1) at the point $P(t,x)\in\Pi\setminus Q_{(0)}$ characterize functions $s_1(\beta_i^{(1)},t)$, $s_2(\beta_i^{(2)},x)$:

$$\begin{aligned} s_1\left(\beta_i^{(1)},t\right) &= \begin{cases} |t-\eta|^{\beta_i^{(1)}}, & |t-\eta|\le 1,\\ 1, & |t-\eta|\ge 1,\end{cases}\\ s_2\left(\beta_i^{(2)},x\right) &= \begin{cases} \rho^{\beta_i^{(2)}}(x), & \rho(x)\le 1,\\ 1, & \rho(x)\ge 1,\end{cases}\end{aligned} \tag{3}$$

where $\rho(x)=\inf_{z\in\bar{\Omega}}|x-z|$, $\beta_i^{(\nu)}\in(-\infty,\infty)$, $\beta^{(\nu)}=(\beta_1^{(\nu)},\cdots,\beta_n^{(\nu)})$, $\nu\in\{1,2\}$, $\beta=(\beta^{(1)},\beta^{(2)})$.

Let us denote that $l,\alpha,q^{(1)},q^{(2)},\gamma^{(1)},\gamma^{(2)},\mu_j^{(1)},\mu_j^{(2)}$ are real numbers, $q^{(\nu)}\ge 0,\gamma^{(\nu)}\ge 0,l\ge 0,\mu_j^{(\nu)}\ge 0,j\in\{0,1,\cdots,n\}$, $[l]$ is the integer part of l, $\{l\}=l-[l]$, $x^{(1)}=(x_1^{(1)},\cdots,x_i^{(1)},\cdots,x_n^{(1)})$, $x^{(2)}=(x_1^{(1)},\cdots,x_{i-1}^{(1)},x_i^{(2)},x_{i+1}^{(1)},\cdots,x_n^{(1)})$, $P(t,x)$, $P_1(t^{(1)},x^{(1)})$, $P_2(t^{(2)},x^{(1)})$, $R_i(t^{(2)},x^{(2)})$ are arbitrary points of the domain $Q^{(k)}$, $i\in\{1,2,\cdots,n\}$.

Let D be an arbitrary closed domain in R^n, $Q^{(k)}=[t_k,t_{k+1})\times D$, $\bar{Q}^{(k)}\subset\Pi^{(k)}=[t_k,t_{k+1})\times R^n$.

We define the functional space in which we study problem (1) and (2).

$H^l(\gamma;\beta;q;\Pi)$ is a set of functions $u(t,x)$ of space $L_1(\Pi)$, which are having continuous partial derivatives in $Q^{(k)}\setminus Q_{(0)}$ of the form $\partial_t^i\partial_x^r u$, $2i+|r|\le[l]$ and a finite value of the norm

$$\begin{aligned}
\|u;\gamma;\beta;q;\Pi\|_l &= \sup_k\left\{\sup_{\bar{Q}^{(k)}}|u|\right\}\equiv\|u,Q\|_0,\\
\|u;\gamma;\beta;q;\Pi\|_l &= \sup_k\left\{\sum_{2i+|r|\le[l]}\left\|u;\gamma;\beta;q,Q^{(k)}\right\|_{2i+|r|}+\left\langle u;\gamma;\beta;q;Q^{(k)}\right\rangle_l\right\}\\
&= \sup_k\left\{\sum_{2i+|r|\le[l]}\sup_{P\in\bar{Q}(k)}\left[s_1\left(q^{(1)}+(2i+|r|)\gamma^{(1)},t^{(1)}\right)\times s_2\left(q^{(2)}+2i\gamma^{(2)},x\right)\left|\partial_t^i\partial_x^r u(P)\right|\prod_{j=1}^n s_1\right.\right.\\
&\quad\cdot\left.\left(-r_j\beta_j^{(1)},t\right)s_2\left(r_j\left(\gamma^{(2)}-\beta_j^{(2)},x\right)\right]+\sup_k\sum_{2i+|r|=[l]}\left\{\sum_{\nu=1}^n\sup_{(P_2,R_\nu)\subset\bar{Q}^{(k)}}\left[s_1\left(q^{(1)}+[l]\gamma^{(1)},t^{(2)}\right)s_2\right.\right.\\
&\quad\cdot\left(q^{(2)}+2i\gamma^{(2)},\tilde{x}\right)\times\prod_{j=1}^n s_1\left(-r_j\beta_j^{(1)},t^{(2)}\right)s_2\left(r_j\left(\gamma^{(2)}-\beta_j^{(2)}\right),\tilde{x}\right)\Big)\left|\partial_t^i\partial_x^r u(P_2)-\partial_t^i\partial_x^r u(R_\nu)\right|\\
&\quad\times\left|x_\nu^{(1)}-x_\nu^{(2)}\right|^{-\{l\}}s_1\left(\{l\}\beta_\nu^{(1)},t^{(2)}\right)s_2\left(\{l\}\left(\gamma^{(2)}-\beta_\nu^{(2)}\right),\tilde{x}\right)+\sup_{(P_1,P_2)\subset\bar{Q}^{(k)}}\left[s_1\left(q^{(1)}+l\gamma^{(1)},\tilde{t}\right)s_2\right.\\
&\quad\cdot\left.\left.\left.\left(q^{(2)}+(2i+\{l\})\gamma^{(2)},x^{(1)}\right)\times\prod_{j=1}^n s_1\left(-r_j\beta_j^{(1)},\tilde{t}\right)s_2\left(r_j\left(\gamma^{(2)}-\beta_j^{(2)}\right),x^{(1)}\right)\left|t^{(1)}-t^{(2)}\right|^{-\{1/2\}}\left|\partial_t^i\partial_x^r u(P_1)-\partial_t^i\partial_x^r u(P_2)\right|\right]\right\}.
\end{aligned} \tag{4}$$

Marked here, $|r|=r_1+r_2+\cdots+r_n$, $s_1(q,\tilde{t})=\min(s_1(q,t^{(1)}),s_1(q,t^{(2)}))$, $s_2(q,\tilde{x})=\min(s_2(q,x^{(1)}),s_2(q,x^{(2)}))$.

We assume that the initial problems (1) and (2) satisfy the following conditions:

(a) For the arbitrary vector $\xi=(\xi_1,\cdots,\xi_n)$, $\forall(t,x)\in\Pi\setminus Q_{(0)}$, the following inequality

$$\begin{aligned} c_1|\xi|^2\le\sum_{i=1}^n s_1\left(\beta_i^{(1)},t\right)s_1\left(\beta_j^{(1)},t\right)s_2\left(\beta_i^{(2)},x\right)s_2\\ \cdot\left(\beta_j^{(2)},x\right)A_{ij}(t,x)\xi_i\xi_j\le c_2|\xi|^2\end{aligned} \tag{5}$$

is true, where c_1,c_2 are fixed positive constants and $s_1(\mu_i^{(1)},t)s_2(\mu_i^{(2)},x)A_i\in H^\alpha(\gamma;\beta;0;\Pi)$, $s_1(\mu_0^{(1)},t)s_2(\mu_0^{(2)},x)A_0\in H^\alpha$

$(\gamma;\beta;0;\Pi)$, $A_0 \geq -a$, $a>0$, $s_1(\beta_i^{(1)},t)s_1(\beta_j^{(1)},t)s_2(\beta_i^{(2)},x)\times$ $\times s_2(\beta_j^{(2)},x)A_{ij} \in H^{\alpha}(\gamma;\beta;0;\Pi)$, $\gamma^{(\nu)} = \max\{\max_i(1+\beta_i^{(\nu)})$ $,\max_i(\mu_i^{(\nu)}-\beta_i^{(\nu)}),\mu_0^{(\nu)}/2\}$, $\nu\in\{1,2\}$.

(b) Functions $f\in H^{\alpha}(\gamma;\beta;\mu_0;\Pi)$, $\varphi_0 \in H^{2+\alpha}(\tilde{\gamma};\tilde{\beta};0;R^n)$, $\varphi_\lambda \in H^{2+\alpha}(\tilde{\gamma};\tilde{\beta};\mu_0;\Pi\cap(t=t_\lambda))$, $\tilde{\gamma}=(0;\gamma^{(2)})$, $\tilde{\beta}=(0;\beta^{(2)})$

Let us formulate the main result of the paper.

Theorem 1. *Let conditions (a) and (b) be satisfied for problem (1) and (2). Then, there exists a unique solution of problem (1) and (2) in the space $H^{2+\alpha}(\gamma;\beta;0;\Pi)$, and the following estimate is correct:*

$$\|u;\gamma;\beta;0;\Pi\|_{2+\alpha} \leq c \sup_{k\in(0,1,\cdots,N)} \left(\|\varphi_k;\gamma,\beta,0;\Pi\cap(t=t_k)\|_{2+\alpha} + \|f;\gamma,\beta,\mu_0;\Pi_k\|_\alpha\right). \quad (6)$$

To study problem (1) and (2), we construct a sequence of solutions of problems with a smooth coefficient limit value of which is the solution of problem (1) and (2).

3. Evaluation of Solutions of Problems with Smooth Coefficients

Let $\Pi_m^{(k)} = \Pi^{(k)} \cap \{(t,x)\in\Pi^{(k)} \mid s_1(1,t)\geq m_1^{-1}, s_2(1,x)\geq m_2^{-1}\}$, $m=(m_1,m_2)$, $m_1>1$, $m_2>1$ be a sequence of domains that for $m_1\to\infty$, $m_2\to\infty$ converges to $\Pi^{(k)}$.

In the domain Π, we consider the problem of finding the functions $u_m(t,x)$ that satisfy the equations

$$(L_1u_m)(t,x) = \left(\partial_t - \sum_{ij=1}^n a_{ij}(t,x)\partial_{x_i}\partial_{x_j} + \sum_{i=1}^n a_i(t,x)\partial_{x_i} + a_0(t,x)\right)u_m(t,x) = f_m(t,x), \quad (7)$$

and the time variable t condition

$$u_m(t_k+0,x) = \varphi_m^{(k)}(x), \quad x\in Q_k\cap(t=t_k). \quad (8)$$

Here, the coefficients a_{ij}, a_i, a_0, and functions f_m, $\varphi_m^{(k)}$, into the domains $\Pi_m^{(k)}$ coincide with A_{ij}, A_i, A_0, f, φ_k, respectively, and in the domains $\Pi\setminus\Pi_m^{(k)}$ are continuous prolongations of coefficients A_{ij}, A_i, A_0 and functions f, φ_k from domains $\Pi_m^{(k)}$ into domains $\Pi\Pi_m^{(k)}$ with preservation of their smoothness and norm ([9], p. 82).

To solve problem (7) and (8), we have a correct theorem.

Theorem 2. *Let $u_m(t,x)$ be the classical solutions of problem (7) and (8) in the domain Π and let conditions (a) and (b) be satisfied. Then, for $u_m(t,x)$, the following estimate*

$$\left\|u_m;\Pi^{(k)}\right\|_0 \leq \left\|\varphi_m^{(k)};\Pi^{(k)}\cap(t=t_k)\right\|_0 + \left\|f_m a_0^{-1}\Pi^{(k)}\right\|_0 \quad (9)$$

is true.

Proof. Let $\max_{\bar{Q}^{(k)}} u_m(t,x) = u_m(M_1)$. If $M_1\in Q^{(k)}$, then at the point M_1, the following correlations

$$\partial_t u_m(M_1)\geq 0,$$
$$\partial_{x_i}u_m(M_1)=0, \quad (10)$$
$$\sum_{i,j=1}^n a_{ij}(M_1)\partial_{x_i}\partial_{x_j}u_m(M_1)\leq 0$$

are true and equation (7) is satisfied. Considering (10) and equation (7) at the point M_1, the inequality

$$u_m(M_1) \leq \sup_{\bar{Q}^{(k)}}\left(fa_0^{-1}\right) \quad (11)$$

is correct.

Let $\min_{\bar{Q}^{(k)}} u_m(t,x) = u_m(M_2)$. If $M_2\in Q^{(k)}$, then at the point M_2, the following correlations

$$\partial_t u_m(M_2)\leq 0,$$
$$\partial_{x_i}u_m(M_2)=0, \quad (12)$$
$$\sum_{i,j=1}^n a_{ij}(M_2)\partial_{x_i}\partial_{x_j}u_m(M_2)\geq 0$$

are true and equation (7) is satisfied. Considering correlation (12) and equation (7) at the point M_2, we have

$$u_m(M_2) \geq \inf_{\bar{Q}^{(k)}}\left(fa_0^{-1}\right). \quad (13)$$

In the case of $M_1\in Q^{(k)}\cap(t=t_k)$ or $M_2\in Q^{(k)}\cap(t=t_k)$ from condition (8), we obtain

$$|u_m| \leq \left\|\varphi_m^{(k)};Q^{(k)}\cap(t=t_k)\right\|_0. \quad (14)$$

Considering inequality (11), (13), and (14), we obtain

$$\left\|u_m;\Pi^{(k)}\right\| \leq \left\|f_m a_0^{-1};\Pi^{(k)}\right\|_0 + \left\|\varphi_m^{(k)};\Pi^{(k)}\cap(t=t_k)\right\|_0. \quad (15)$$

The theorem is proved.

Now, we find estimates of the derivatives of solutions $u_m(t,x)$. In the space $C^l(\Pi)$, we introduce a norm $\|u_m;\gamma;\beta;q;\Pi\|_l$ which is equivalent, at fixed m_1, m_2, to the Hölder norm, which is defined by the same way as the norm $\|u;\gamma;\beta;q;\Pi\|_l$; only, instead of functions $s_1(q^{(1)},t)$ and $s_2(q^{(2)},x)$, we take $d_1(q^{(1)},t)$ and $d_2(q^{(2)},x)$, respectively,

$$d_1\left(q^{(1)},t\right)=\begin{cases}\max\left(s_1\left(q^{(1)},t\right),m_1^{-q^{(1)}}\right), & q^{(1)}\ge 0,\\ \min\left(s_1\left(q^{(1)},t\right),m_1^{-q^{(1)}}\right), & q^{(1)}<0,\end{cases}$$
$$d_2\left(q^{(2)},x\right)=\begin{cases}\max\left(s_2\left(q^{(2)},x\right),m_2^{-q^{(2)}}\right), & q^{(2)}\ge 0,\\ \min\left(s_2\left(q^{(2)},x\right),m_2^{-q^{(2)}}\right), & q^{(2)}<0.\end{cases} \quad (16)$$

Theorem 3. *Let conditions (a) and (b) be satisfied. Then, for the solution of problem (7) and (8), the estimate*

$$\left\|u_m;\gamma;\beta;0;\Pi^{(k)}\right\|_{2+\alpha}\le c\Big(\left\|f;\gamma;\beta;\mu_0;\Pi^{(k)}\right\|_{\alpha}+\left\|\varphi_k;\gamma;\beta;0;\Pi^{(k)}\cap(t=t_{k-1})\right\|_{2+\alpha}\Big) \quad (17)$$

is true.

Proof. Using the definition of the norm and interpolation inequalities from [9], we have

$$\left\|u_m;\gamma;\beta;0;\Pi^{(k)}\right\|_{2+\alpha}\le(1+\varepsilon^{\alpha})\left\langle u_m;\gamma;\beta;0;\Pi^{(k)}\right\rangle_{2+\alpha}+C(\varepsilon)\left\|u_m;\Pi^{(k)}\right\|_0, \quad (18)$$

where ε is an arbitrary real number $\varepsilon\in(0,1)$. Hence, it is sufficient to estimate the seminorm $\langle u_m;\gamma;\beta;0;\Pi^{(k)}\rangle_{2+\alpha}$. As follows from the definition of seminorm, there exist in $\Pi^{(k)}$ points P_1,P_2,R_ν, for which one of the inequalities

$$\frac{1}{2}\left\|u_m;\gamma;\beta;0;\Pi^{(k)}\right\|_{2+\alpha}\le E_\mu,\quad \mu\in\{1,2\}, \quad (19)$$

is true, where

$$E_1\equiv\sum_{2i+|r|=2}\left\{\sum_{\nu=1}^{n}d_1\left(2\gamma^{(1)},t^{(2)}\right)d_2\left(2i\gamma^{(2)},\tilde{x}\right)\prod_{i=1}^{n}d_1\cdot\left(-r_i\beta_i^{(1)},t^{(2)}\right)\times d_2\left(r_i\left(\gamma^{(2)}-\beta_i^{(2)}\right),\tilde{x}\right)\left|\partial_t^i\partial_x^r u_m(P_2)-\partial_t^i\partial_x^r u_m(R_\nu)\right|\left|x_\nu^{(1)}-x_\nu^{(2)}\right|^{-\alpha}\times d_1\left(\alpha\beta_\nu^{(1)},t^{(2)}\right)d_2\cdot\left(\alpha\left(\gamma^{(2)}-\beta_\nu^{(2)}\right),\tilde{x}\right),\right.$$

$$E_2\equiv\sum_{2i+|r|=2}d_1\left((2+\alpha)\gamma^{(1)},\tilde{t}\right)d_2\left(r_i\left(\gamma^{(2)}-\beta_i^{(2)}\right),x^{(1)}\right)\prod_{i=1}^{n}d_1\cdot\left(-r_i\beta_i^{(1)},\tilde{t}\right)\times d_2\left(r_i\left(\gamma^{(2)}-\beta_i^{(2)}\right),x^{(1)}\right)\left|t^{(1)}-t^{(2)}\right|^{-(\alpha/2)}\cdot\left|\partial_t^i\partial_x^r u_m(P_1)-\partial_t^i\partial_x^r u_m(P_2)\right|,\quad 2s+|r|=2. \quad (20)$$

If $|x_\nu^{(1)}-x_\nu^{(2)}|\ge(\varepsilon n^{-1}/4)d_1(\gamma^{(1)},\tilde{t})d_2(\gamma^{(2)}-\beta_\nu^{(2)},\tilde{x})\equiv T_1$, and ε_1 is an arbitrary real number from (0,1), then

$$E_1\le 2\varepsilon_1^{-\alpha}\left\|u_m;\gamma;\beta;0;\Pi^{(k)}\right\|_2. \quad (21)$$

If $|t^{(1)}-t^{(2)}|\ge(\varepsilon_1^2/16)d_1(2\gamma^{(1)},\tilde{t})d_2(2\gamma^{(2)},\tilde{x})\equiv T_2$, then

$$E_2\le 2\varepsilon_1^{-\alpha}\left\|u_m;\gamma;\beta;0;\Pi^{(k)}\right\|_2. \quad (22)$$

Applying the interpolation inequalities to (21) and (22), we find

$$E_\mu\le\varepsilon^{\alpha}\left\|u_m;\gamma;\beta;0;\Pi^{(k)}\right\|_{2+\alpha}+c(\varepsilon)\left\|u_m;\Pi^{(k)}\right\|_0. \quad (23)$$

Let $|x_\nu^{(1)}-x_\nu^{(2)}|\le T_1$, and let $|t^{(1)}-t^{(2)}|\le T_2$. We assume that $|x_\nu^{(1)}-\xi_\nu|\ge 4T_1$, $\xi\in\partial D$ and

$$d_2\left(\gamma^{(2)},\tilde{x}\right)=\min\left(d_2\left(\gamma^{(2)},x^{(1)}\right),d_2\left(\gamma^{(2)},x^{(2)}\right)\right)\equiv d_2\left(\gamma^{(2)},x^{(1)}\right),$$
$$d_1\left(\gamma^{(2)},\tilde{t}\right)=\min\left(d_1\left(\gamma^{(1)},t^{(1)}\right),d_1\left(\gamma^{(1)},t^{(2)}\right)\right)=d_1\left(\gamma^{(1)},t^{(1)}\right), \quad (24)$$

$P_1(t^{(1)},x^{(1)})\in\Pi^{(k)}$, $k\in\{0,1,\cdots,N\}$. In the domain $Q^{(k)}$, we write the problem (7) and (8) in the form

$$\left[\partial_t-\sum_{ij=1}^{n}a_{ij}(P_1)\partial_{x_i}\partial_{x_j}\right]u_m=\sum_{ij=1}^{n}\left[a_{ij}(P)-a_{ij}(P_1)\right]\partial_{x_i}\partial_{x_j}u_m-\sum_{i=1}^{n}a_i(P)\partial_{x_i}u_m-a_0(P)u_m+f_m(t,x)=F_m(t,x), \quad (25)$$

$$u_m(t_k+0,x)=\Phi_m^{(k)}(t_k,x). \quad (26)$$

Let $\Pi_\tau^{(k)}$ be a domain from $Q^{(k)}$, $\Pi_\tau^{(k)}=\{(t,x)\in Q^{(k)},\| x_\nu-x_\nu^{(1)}|\le\tau T_1,\nu\in\{1,2\},|t-t^{(1)}|\le\tau^2T_2\}$. Performing the substitution,

$$u_m(t,x)=v_m(t,y),\quad x_\nu=d_1\left(\beta_\nu^{(1)},t^{(1)}\right)d_2\left(\beta_\nu^{(2)},x^{(1)}\right)y_\nu, \quad (27)$$

in problem (25) and (26), we obtain

$$\begin{aligned}(L_2 v_m)(t,y) \equiv \Bigg[\partial_t - \sum_{ij=1}^{n} a_{ij}(P_1) d_1\left(\beta_i^{(1)}, t^{(1)}\right) d_1\left(\beta_j^{(1)}, t^{(1)}\right) \\ \times d_2\left(\beta_i^{(2)}, x^{(1)}\right) d_2\left(\beta_j^{(2)}, x^{(1)}\right) \partial_{y_i}\partial_{y_j}\Bigg] v_m \\ = F_m(t,\tilde{y}), \\ v_m(t_k+0,\tilde{y}) = \Phi_m^{(k)}(t_k,\tilde{y}),\end{aligned} \tag{28}$$

where $\tilde{y} = (d_1(-\beta_1^{(1)}, t^{(1)}) d_2(-\beta_1^{(2)}, x^{(1)}) x_1, \cdots, d_1(-\beta_n^{(1)}, t^{(1)}) d_2(-\beta_n^{(2)}, x^{(1)}) x_n)$.

We denote

$$\begin{aligned} y_\nu^{(1)} &= d_1\left(-\beta_\nu^{(1)}, t^{(1)}\right) d_2\left(-\beta_\nu^{(2)}, x^{(1)}\right) x_\nu^{(1)}, \\ V_\tau^{(k)} &= \left\{(t,y), \left|t - t^{(1)}\right| \le \tau^2 T_2, \left|y_\nu - y_\nu^{(1)}\right| \le \frac{\tau}{n}\sqrt{T_2}\right\},\end{aligned} \tag{29}$$

and choose a thrice differentiable function $\psi(t,y)$, which satisfies the conditions

$$\psi(t,y) = \begin{cases} 1, & (t,y) \in V_{1/2}^{(k)}, 0 \le \psi(t,y) \le 1, \\ 0, & (t,y) \in V_{3/4}^{(k)}, \left|\partial_t^i \partial_x^r \psi\right| \le c_{ri} d_1\left(-(2i+|r|)\gamma^{(1)}; t^{(1)}\right) \times \times d_2\left(-(2i+|r|)\gamma^{(2)}, x^{(1)}\right). \end{cases} \tag{30}$$

We denote the function $Z_m(t,y) = v_m(t,y)\psi(t,y)$ which is a solution of the Cauchy problem

$$\begin{aligned}(L_3 Z_m)(t,y) = \sum_{ij=1}^{n} a_{ij}(P_1) d_1\left(\beta_i^{(1)}, t^{(1)}\right) d_1\left(\beta_j^{(1)}, t^{(1)}\right) d_2 \\ \cdot \left(\beta_i^{(2)}, x^{(1)}\right) d_2\left(\beta_j^{(2)}, x^{(1)}\right) \\ \times \left[\partial_{y_i} v_m \partial_{y_j}\psi + \partial_{y_j} v_m \partial_{y_i}\psi + v_m \partial_{y_i}\partial_{y_j}\psi\right] \\ - v_m \partial_t \psi + \psi F_m(t,\tilde{y}) \\ = F_m^{(1)}(t,\tilde{y},\psi),\end{aligned} \tag{31}$$

$$Z_m(t_k+0,x) = \Phi_m^{(k)}(t_k,\tilde{y})\psi(t_k,y). \tag{32}$$

On the basis of Theorem 5.3 from ([2], p. 364), for the solution of problem (31) and (32) and arbitrary points $(M_1, M_2) \subset V_{1/2}^{(k)}$ inequalities

$$\begin{aligned} d^{-\alpha}(M_1,M_2)\left|\partial_t^i\partial_x^r v_m(M_1) - \partial_t^i\partial_x^r v_m(M_2)\right| \\ \le c\left(\left\|F_m^{(1)}\right\|_{C^\alpha(V_{3/4})} + \left\|\psi\Phi_m^{(k)}\right\|_{C^{2+\alpha}(V_{3/4}\cap(t=t_k))}\right)\end{aligned} \tag{33}$$

are true, where $2i+|r|=2$, $d(M_1,M_2)$ is the parabolic distance between the points M_1 and M_2. With regard to the properties of function $\psi(t,y)$, we find estimates of the norms of expressions $\|F_m^{(1)}\|$ and $\|\psi\Phi_m^{(k)}\|$:

$$\begin{aligned}\left\|F_m^{(1)}\right\|_{C^\alpha(V_{3/4})} \le c d_1\left(-(2+\alpha)\gamma^{(1)}, t^{(1)}\right) d_2\left(-(2+\alpha)\gamma^{(2)}, x^{(1)}\right) \\ \times \left(\left\|v_m; V_{3/4}^{(k)}\right\|_0 + \left\|v_m; \gamma; 0; 0; V_{3/4}^{(k)}\right\|_2 \right. \\ \left. + \left\|F_m; \gamma; 0; 2\gamma; V_{3/4}^{(k)}\right\|_\alpha\right),\end{aligned} \tag{34}$$

$$\begin{aligned}\left\|\psi\Phi_m^{(k)}\right\|_{C^{2+\alpha}(V_{3/4}\cap(t=t_k))} \le c d_1\left(-(2+\alpha)\gamma^{(1)}, t^{(1)}\right) d_2 \\ \cdot \left(-(2+\alpha)\gamma^{(2)}, x^{(1)}\right) \\ \times \left\|\Phi_m^{(k)}; \gamma; 0; V_{3/4}^{(k)}\right\|_{2+\alpha}.\end{aligned} \tag{35}$$

The definition of the space $H^l(\gamma;\beta;q;Q)$ implies the satisfaction of inequalities

$$\begin{aligned} c_1\left\|v_m; \gamma; 0; 0; V_{3/4}^{(k)}\right\|_l \le \left\|u_m; \gamma; \beta; 0; \Pi_{3/4}^{(k)}\right\|_l \\ \le c_2\left\|v_m; \gamma; 0; 0; V_{3/4}^{(k)}\right\|_l.\end{aligned} \tag{36}$$

Substituting (34) and (35) into (33) and returning to the variables (t, x), we obtain inequalities

$$E_\mu \le c_1 \Big(\left\| F_m ; \gamma ; \beta ; 2\gamma ; \Pi_{3/4}^{(k)} \right\|_\alpha + \left\| \Phi_m^{(k)} ; \gamma ; \beta ; 0 ; \Pi_{3/4}^{(k)} \cap (t = t_k) \right\|_{2+\alpha} + \left\| u_m ; \Pi_{3/4}^{(k)} \right\|_0 + \left\| u_m ; \gamma ; \beta ; 0 ; \Pi_{3/4}^{(k)} \right\|_2 \Big). \quad (37)$$

Given the interpolation inequalities and estimates of the norm of each additive of the expressions F_m, $\Phi_m^{(k)}$, we obtain the inequalities

$$E_\mu \le \left(\varepsilon^\alpha (n+2) + \varepsilon_1 C n^2 \right) \left\| u_m ; \gamma ; \beta ; 0 ; \Pi_{3/4}^{(k)} \right\|_{2+\alpha} + c_2 \Big(\left\| u_m ; \Pi_{3/4}^{(k)} \right\|_0 + c_3 \Big(\left\| F_m ; \gamma ; \beta ; 2\gamma ; \Pi_{3/4}^{(k)} \right\|_\alpha + \left\| \Phi_m^{(k)} \gamma ; \beta ; 0 ; \Pi_{3/4}^{(k)} \cap (t = t_k) \right\|_{2+\alpha} \Big). \quad (38)$$

Using inequalities (23) and (37) and choosing ε_1 and ε sufficiently small, we obtain the estimate

$$\left\| u_m ; \gamma ; \beta ; 0 ; \Pi^{(k)} \right\|_{2+\alpha} \le c \Big(\left\| F_m ; \gamma ; \beta ; 2\gamma ; \Pi^{(k)} \right\|_\alpha + \left\| \Phi_m^{(k)} \gamma ; \beta ; 0 ; \Pi^{(k)} \cap (t = t_k) \right\|_{2+\alpha} \Big). \quad (39)$$

Given the values of the expression $F_m(t, x)$ and $\Phi_m^{(k)}(t_k, x)$, for $k = 1, 2, \cdots, N$, we have

$$\left\| F_m ; \gamma ; \beta ; 2\gamma ; \Pi^{(k)} \right\|_\alpha \le c \Big(\left\| f_m ; \gamma ; \beta ; \mu_0 ; \Pi^{(k)} \right\|_\alpha,$$
$$\left\| \Phi_m^{(k)} ; \gamma ; \beta ; 0 ; \Pi \cap (t = t_k) \right\|_{2+\alpha} \le c \left\| \varphi_m^{(k)} ; \gamma ; \beta ; 0 ; \Pi \cap (t = t_k) \right\|_{2+\alpha}. \quad (40)$$

Because

$$\left\| f_m ; \gamma ; \beta ; \mu_0 ; \Pi^{(k)} \right\|_\alpha \le c \left\| f ; \gamma ; \beta ; \mu_0 ; \Pi^{(k)} \right\|_\alpha,$$
$$\left\| \varphi_m^{(k)} ; \gamma ; \beta ; 0 ; \Pi^{(k)} \cap (t = t_k) \right\|_{2+\alpha} \le c \left\| \varphi_k ; \gamma ; \beta ; 0 ; \Pi^{(k)} \cap (t = t_k) \right\|_{2+\alpha}, \quad (41)$$

then, given the estimate (9) and inequalities (39) and (40) for $k = 0, 1, \cdots, N$, the estimate (17) is true. The theorem is proved.

Proof of Theorem 4. The right-hand side of inequality (17) is independent of m, and the sequences

$$\left\{ u_m^{(0)} \right\} \equiv \{ u_m \},$$
$$\left\{ u_m^{(1)} \right\} \equiv \left\{ d_1 \left(\gamma^{(1)} - \beta_i^{(1)}, t \right) d_2 \left(\gamma^{(2)} - \beta_i^{(2)}, x \right) \partial_{x_i} u_m \right\},$$
$$\left\{ u_m^{(2)} \right\} \equiv \left\{ d_1 \left(2\gamma^{(1)}, t \right) d_2 \left(2\gamma^{(2)}, x \right) \partial_t u_m \right\},$$
$$\left\{ u_m^{(3)} \right\} = \left\{ d_1 \left(\gamma^{(1)} - \beta_i^{(1)}, t \right) d_2 \left(\gamma^{(2)} - \beta_i^{(2)}, x \right) d_1 \cdot \left(\gamma^{(1)} - \beta_j^{(1)}, t \right) d_2 \left(\gamma^{(2)} - \beta_j^{(2)}, x \right) \partial_{x_i} \partial_{x_j} u_m \right\} \quad (42)$$

are uniformly bounded and equicontinuous in $Q^{(k)}$. According to Arcel's theorem, there exist subsequences $\{u_{m(j)}^{(\mu)}\}$ which are uniformly convergent to $\{u_0^{(\mu)}\}$ in $Q^{(k)}$ for $m(j) \to \infty$, $\mu \in \{0, 1, 2, 3\}$. Passing to the limit as $m(j) \to \infty$ in problem (7) and (8), we obtain that $u(t, x) = u_0^{(0)}$ is the unique solution of problem (1) and (2) and $u \in H^{2+\alpha}(\gamma ; \beta ; 0 ; \Pi)$. The theorem is proved.

4. Conclusions

The necessary and sufficient conditions for the existence of the unique solution of a multipoint problem for parabolic equations with degeneration are established. Estimates of derivatives of the solution of the problem in the Hölder spaces with power weight are found. The order of the degree weight depends on the power of the degree features of the coefficients of the equation.

Data Availability

The article contains theoretical material. This paper has links to articles and textbooks from other researchers. The article does not use any data.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this article.

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