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# NONLOCAL MULTIPOINT (IN TIME) PROBLEM WITH OBLIQUE DERIVATIVE FOR A PARABOLIC EQUATION WITH DEGENERATION

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For a second-order parabolic equation, we consider a multipoint (in time) problem with oblique derivative. The coefficients of the equation and boundary condition have power singularities of any order in time and spatial variables on a certain set of points. We establish conditions for the existence and uniqueness of solution of the posed problem in Hölder's spaces with power weight.

In the last decades, special attention is given to problems for partial differential equations with nonlocal conditions. The interest to problems of this kind is explained both by the needs of the general theory of boundary-value problems and by their wide practical applications (diffusion and vibration processes, salt and moisture transfer in soils, plasma physics, mathematical biology, etc.). It was also shown that nonlocal conditions can be used for the description of solvable extensions of differential operators. Ptashnyk and his colleagues [1, 4, 6, 7], used the metric approach to the investigation of the conditionally well-posed problems with multipoint conditions with respect to the selected variable for linear hyperbolic and typeless equations and also for some classes of parabolic equations with constant coefficients. They proved metric theorems on the lower estimates of small denominators encountered in the solution of the analyzed problems, which yield the well-posedness of these problems for almost all (with respect to the Lebesgue measure) values of the parameters.

In bounded cylindrical domains, the problems with nonlocal and integral conditions with respect to the time variable were studied in [2, 8–10, 14] for parabolic equations with power singularities of any order in the coefficients of equations and in the boundary conditions (with respect to any variables on a certain set of points). The works [3, 11, 13] were devoted to the classical solutions of boundary-value problems with impulsive action for second-order parabolic equations whose coefficients have power singularities with respect to the spatial variables.

In the present work, we investigate a multipoint (in time) problem with oblique derivative for a second-order parabolic equation with power singularities and degenerations of any order in the coefficients of the equation and in boundary condition with respect to any variables on a certain set of points. We also establish conditions for the existence and uniqueness of solution of the formulated problem in Hölder's spaces with power weight.

## 1. Statement of the Problem and Basic Restrictions

Suppose that $D$ is a bounded domain in $\mathbb{R}^n$ with boundary $\partial D$, $\dim D = n$, $\Omega$ is a bounded domain, $\bar{\Omega} \subset D$, $\dim \Omega \le n-1$, $\eta$, $t_0$, $t_1, \ldots, t_N$, and $t_{N+1}$ are fixed positive numbers, $0 \le t_0 < t_1 < \ldots < t_{N+1}$, $t_0 < \eta < t_{N+1}$, $\eta \ne t_\lambda$, $\lambda \in \{1,2,\ldots,N\}$. We denote

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$$Q_{(0)}=\{(t,x):t\in[t_0,t_{N+1}),\,x\in\Omega\}\bigcup\,\{(t,x):t=\eta,x\in D\},$$

$$Q^{(k)}=[t_k,t_{k+1})\times D,\qquad k\in\{0,1,\ldots,N\},$$

and

$$\Gamma^{(k)}=[t_k,t_{k+1})\times\partial D.$$

In the domain $Q=[t_0,t_{N+1})\times D$, we consider a problem of finding a function $u(t,x)$ satisfying, for $t\neq t_k$, $(t,x)\notin Q_{(0)}$, an equation

$$(Lu)(t,x)=\left[\partial_t-\sum_{i,j=1}^{n}A_{ij}(t,x)\partial_{x_i}\partial_{x_j}+\sum_{i=1}^{n}A_i(t,x)\partial_{x_i}+A_0(t,x)\right]u(t,x)=f(t,x),\tag{1}$$

a multipoint (in time $t$) condition

$$u(t_k+0,x)=\varphi_k(x),\qquad x\in D\backslash\bar{\Omega},\qquad k\in\{0,1,\ldots,N\},\tag{2}$$

and a boundary condition

$$\lim_{x\to z\in\partial D}(Bu-g)(t,x)\equiv\lim_{x\to z\in\partial D}\left[\sum_{i=1}^{n}b_i(t,x)\partial_{x_i}u+b_0(t,x)u-g(t,x)\right]=0.\tag{3}$$

The power singularities of coefficients of the differential expressions $L$ and $B$ at point $P(t,x)\in Q\backslash Q_0$ are characterized by functions $s_1(\beta_i^{(1)},t)$ and $s_2(\beta_i^{(2)},x)$:

$$s_1(\beta_i^{(1)},t)=\begin{cases}|t-\eta|^{\beta_i^{(1)}}, & |t-\eta|\le 1,\\ 1, & |t-\eta|\ge 1,\end{cases}\qquad s_2(\beta_i^{(2)},x)=\begin{cases}\rho^{\beta_i^{(2)}}(x), & \rho(x)\le 1,\\ 1, & \rho(x)\ge 1,\end{cases}$$

$$\rho(x)=\inf_{z\in\bar{\Omega}}|x-z|,\qquad \beta_i^{(\nu)}\in(-\infty,\infty),\quad \nu\in\{1,2\},\qquad \beta^{(\nu)}=(\beta_1^{(\nu)},\ldots,\beta_n^{(\nu)}),\qquad \beta=(\beta^{(1)},\beta^{(2)}).$$

We now define the spaces used to study problem (1)–(3). Let $\ell$, $q^{(1)}$, $q^{(2)}$, $\gamma^{(1)}$, $\gamma^{(2)}$, $\mu_j^{(1)}$, $\mu_j^{(2)}$, $\delta^{(1)}$, and $\delta^{(2)}$ be real numbers. Also let $\ell\ge 0$, $[\ell]$ be the integer part of $\ell$, $\{\ell\}=\ell-[\ell]$; $q^{(\nu)}\ge 0$, $\gamma^{(\nu)}\ge 0$, $\mu_j^{(\nu)}\ge 0$, $\delta^{(\nu)}\ge 0$, $\nu\in\{1,2\}$, $j\in\{0,\ldots,n\}$; and let $P(t,x)$, $P_1(t^{(1)},x^{(1)})$, $P_2(t^{(2)},x^{(1)})$, and $R_i(t^{(1)},x^{(2)})$ be arbitrary points from $Q^{(k)}$, $i\in\{1,2,\ldots n\}$, $x^{(1)}=(x_1^{(1)},x_2^{(1)},\ldots,x_n^{(1)})$, $x^{(2)}=(x_1^{(1)},\ldots,x_{i-1}^{(1)},x_i^{(2)},x_{i+1}^{(1)},\ldots,x_n^{(1)})$.

By $H^{\ell}(\gamma;\beta;q;Q)$ we denote the set of functions $u$ with continuous derivatives in $Q^{(k)}\backslash Q_{(0)}$ for $t\neq t_k$ of the form $\partial_t^s\partial_x^r$, (here, $2s+|r|\le[\ell]$, $|r|=r_1+\ldots+r_n$, $r=(r_1,r_2,\ldots,r_n)$ is a multiindex) for which the norm

$$\|u;\gamma;\beta;0;Q\|_0=\sup_k\{\sup_{\bar{Q}^{(k)}}|u|\}\equiv\|u;Q\|_0,$$

$$\|u;\gamma;\beta;q;Q\|_{\ell} = \sup_{k}\left\{\sum_{2s+|r|\le[\ell]}\left\|u;\gamma;\beta;q;Q^{(k)}\right\|_{2s+|r|} + \left\langle u;\gamma;\beta;q;Q^{(k)}\right\rangle_{\ell}\right\}$$

is finite. Here, e.g.,

$$\left\|u;\gamma;\beta;q;Q^{(k)}\right\|_{2s+|r|} \equiv \sup_{P\in\bar{Q}^{(k)}}\Bigg[s_1(q^{(1)}+2s\gamma^{(1)},t)s_2(q^{(2)}+2s\gamma^{(2)},x)$$

$$\times\left|\partial_t^s\partial_x^r u(P)\right|\prod_{i=1}^{n}s_1(r_i(\gamma^{(1)}-\beta_i^{(1)}),t)s_2(r_i(\gamma^{(2)}-\beta_i^{(2)}),x)\Bigg],$$

$$\left\langle u;\gamma,\beta,q;Q^{(k)}\right\rangle_{\ell} \equiv \sum_{2s+|r|=[\ell]}\Bigg\{\sum_{\nu=1}^{n}\Bigg[\sup_{(P_2,R_\nu)\subset\bar{Q}_k}\Bigg[s_1(q^{(1)}+2s\gamma^{(1)},t^{(2)})s_2(q^{(2)}$$

$$+2s\gamma^{(2)},\tilde{x})\prod_{i=1}^{n}s_1(r_i(\gamma^{(1)}-\beta_i^{(1)}),t^{(2)})s_2(r_i(\gamma^{(2)}-\beta_i^{(2)}),\tilde{x})$$

$$\times\left|\partial_t^s\partial_x^r u(P_2)-\partial_t^s\partial_x^r u(R_\nu)\right|\left|x_\nu^{(1)}-x_\nu^{(2)}\right|^{-\{\ell\}}$$

$$\times s_1(\{\ell\}(\gamma^{(1)}-\beta_\nu^{(1)}),t^{(2)})s_2(\{\ell\}(\gamma^{(2)}-\beta_\nu^{(2)}),\tilde{x})\Bigg]$$

$$+\sup_{(P_1,P_2)\subset\bar{Q}_k}\Bigg[s_1(q^{(1)}+\ell\gamma^{(1)},\tilde{t})s_2(q^{(2)}+(2s+\{\ell\})\gamma^{(2)},x^{(1)})$$

$$\times\prod_{i=1}^{n}s_1(-r_i\beta_i^{(1)},\tilde{t})s_2(r_i(\gamma^{(2)}-\beta_i^{(2)}),x^{(1)})\left|t^{(1)}-t^{(2)}\right|^{-\{\ell/2\}}$$

$$\times\left|\partial_t^s\partial_x^r u(P_1)-\partial_t^s\partial_x^r u(P_2)\right|\Bigg]\Bigg]\Bigg\},$$

$$s_1(q,\tilde{t}) = \min\{s_1(q,t^{(1)}),s_1(q^{(1)},t^{(2)})\},\quad \text{and}\quad s_2(q,\tilde{x}) = \min\{s_2(q,x^{(1)}),s_2(q,x^{(2)})\}.$$

Suppose that the following conditions are satisfied for problem (1)–(3):

(1°). For any vector $\xi=(\xi_1,\dots,\xi_n)$ and $\forall(t,x)\in Q\backslash Q_{(0)}$, the inequality

$$\pi_1|\xi|^2 \le \sum_{i,j=1}^{n}A_{ij}(t,x)s_1(\beta_i^{(1)},t)s_1(\beta_j^{(1)},t)s_2(\beta_i^{(2)},x)s_2(\beta_j^{(2)},x)\xi_i\xi_j \le \pi_2|\xi|^2 \quad (4)$$

is true. Here, $\pi_1$ and $\pi_2$ are fixed positive constants,

$$|\xi|^2 = \sum_{i=1}^{n} \xi_i^2, \quad s_1(\mu_i^{(1)},t)s_2(\mu_i^{(2)},x)A_i \in H^{\alpha}(\gamma;\beta;0;Q), \quad i \in \{0,\ldots,n\},$$

$$A_0 \geq -a, \quad a > 0, \quad s_1(\delta^{(1)},t)s_2(\delta^{(2)},x)b_0 \in H^{1+\alpha}(\gamma;\beta;0;Q),$$

$$s_1(\beta_i^{(1)},t)s_1(\beta_j^{(1)},t)s_2(\beta_i^{(2)},x)s_2(\beta_j^{(2)},x)A_{ij} \in H^{\alpha}(\gamma;\beta;0;Q),$$

$$s_1(\beta_i^{(1)},t)s_2(\beta_i^{(2)},x)b_i \in H^{1+\alpha}(\gamma;\beta;0;Q),$$

the vectors $b^{(s)} = \{b_1^{(s)},\ldots,b_n^{(s)}\}$, where

$$b_i^{(s)} = s_1(\beta_i^{(1)},t)s_2(\beta_i^{(2)},x)b_i,$$

and $e = \{e_1,\ldots,e_n\}$, where

$$e_i = b_i\Big(\sum_{k=1}^{n} b_k^2\Big)^{-1/2},$$

form an angle smaller than $\frac{\pi}{2}$ with the direction of outer normal $n$ to $\partial D$ at the point $P(t,x) \in \Gamma$,

$$\Gamma = [t_0, t_{N+1}) \times \partial D, \quad b_0(t,x)\big|_{\Gamma} > 0.$$

(2°).

$$f \in H^{\alpha}(\gamma;\beta;\mu_0;Q), \quad \varphi_k \in H^{2+\alpha}(\tilde{\gamma};\tilde{\beta},0;Q \cap (t = t_k)), \quad \tilde{\gamma} = (0,\gamma^{(2)}), \quad \tilde{\beta} = (0,\beta^{(2)}),$$

$$\lim_{x \to z \in \partial D} (B\varphi_k - g)(t_k,x) = 0, \quad k \in \{0,1,\ldots,N\}, \quad g \in H^{1+\alpha}(\gamma;\beta;\delta;Q^{(k)}),$$

$$\gamma^{(\nu)} = \max\left\{\max_i (1+\beta_i^{(\nu)}), \max_i (\gamma^{(\nu)} - \beta_i^{(\nu)}), \frac{\mu_0^{(\nu)}}{2}, \delta^{(\nu)}\right\}, \quad \nu \in \{1,2\}.$$

The following theorem is true:

**Theorem 1.** *Suppose that conditions (1°) and (2°) are satisfied for problem (1)–(3). Then there exists a unique solution of problem (1)–(3) from the space* $H^{2+\alpha}(\gamma;\beta;0;Q)$, *and the inequality*

$$\|u;\gamma,\beta,0;Q\|_{2+\alpha} \leq c \sup_k \Big(\|\varphi_k;Q \cap (t = t_k)\|_{2+\alpha} + \big\|f;Q^{(k)}\big\|_{\alpha} + \big\|g;Q^{(k)}\big\|_{1+\alpha}\Big) \tag{5}$$

*is true.*

*If* $f \in H^{\alpha}(\gamma;\beta;0;Q)$, $g \in H^{1+\alpha}(\gamma;\beta;0;Q^{(k)})$, *and conditions (1°) and (2°) are satisfied for problem (1)–(3), then the unique solution of problem (1)–(3) in the domain* $Q^{(k)}$ *is determined by the following Stieltjes integrals with Borel measure* $G(t,x;Z)$:

$$u(t,x) = \int\limits_{Q^{(k)}} G_1^{(k)}(t,x;d\tau,d\xi) f(\tau,\xi) + \int\limits_{D} G_2^{(k)}(t,x;d\xi)\varphi_k(\xi)$$

$$+ \int\limits_{\Gamma^{(k)}} G_3^{(k)}(t,x;d\tau,d_\xi S) g(\tau,\xi), \tag{6}$$

*whose components* $G_1^{(k)}$, $G_2^{(k)}$, *and* $G_3^{(k)}$ *satisfy the inequalities*

$$\left| \int\limits_{Q^{(k)}} G_1^{(k)}(t,x,d\tau,d\xi) \right| \le \left\| A_0^{-1}; Q^{(k)} \right\|_0, \qquad \left| \int\limits_{D} G_2^{(k)}(t,x;d\xi) \right| \le 1,$$

$$\left| \int\limits_{\Gamma^{(k)}} G_3^{(k)}(t,x;d\tau,d_\xi S) \right| \le \left\| b_0^{-1}; Q^{(k)} \right\|_0. \tag{7}$$

To prove Theorem 1, we first establish the correct solvability of boundary-value problems with smooth coefficients. In the set of obtained solutions, we select a convergent subsequence whose limit value is the solution of problem (1)–(3).

## 2. Estimation of the Solutions of Problems with Smooth Coefficients

Let $Q_m^{(k)} = Q^{(k)} \cap \{(t,x) \in Q^{(k)} : s_1(1,t) \ge m_1^{-1}, s_2(1,x) \ge m_2^{-1}, m = (m_1, m_2), m_i > 1, i \in \{1,2\}\}$ be a sequence of domains convergent to $Q^{(k)}$ as $m_i \to \infty$.

In the domain $Q = [t_0, t_{N+1}) \times D$, we consider the problem of finding the solutions of the equation

$$(L_1 u_m)(t,x) \equiv \left[ \partial_t - \sum_{i,j=1}^{n} a_{ij}(t,x) \partial_{x_i} \partial_{x_j} + \sum_{i=1}^{n} a_i(t,x) \partial_{x_i} + a_0(t,x) \right] u_m(t,x) = f_m(t,x) \tag{8}$$

satisfying the multipoint conditions (in the variable $t$):

$$u_m(t_k + 0, x) = \varphi_m^{(k)}(x), \qquad k \in \{0,1,\ldots,N\}, \tag{9}$$

and the following boundary condition on the lateral surface:

$$\lim_{x \to z \in \partial D} (B_1 u_m - g_m)(t,x)$$

$$= \lim_{x \to z \in \partial D} \left[ \sum_{i=1}^{n} h_i(t,x) \partial_{x_i} u_m + h_0(t,x) u_m - g_m(t,x) \right] = 0. \tag{10}$$

Here, the coefficients $a_{ij}$, $a_i$, $a_0$, $h_i$, and $h_0$ and the functions $f_m$, $\varphi_m^{(k)}$, and $g_m$ are determined in the following way: If $(t,x)\in Q_m^{(k)}$, then the coefficients $a_{ij}$, $a_i$, $a_0$, $h_i$, $h_0$ and functions $f_m$, $\varphi_m^{(k)}$, $g_m$ coincide with $A_{ij}$, $A_i$, $A_0$, $f$, $\varphi_k$, and $g$, respectively. In the domains $Q^{(k)}\backslash Q_m^{(k)}$, these coefficients and functions are regarded as continuous extensions of the coefficients $A_{ij}$, $A_i$, $A_0$, $b_i$, and $b_0$ and functions $f$, $\varphi_k$, and $g$ from the domains $Q_m^{(k)}$ into the domains $Q^{(k)}\backslash Q_m^{(k)}$ with preservation of their smoothness and norms [12, p. 82].

The following theorem is true:

**Theorem 2.** *Suppose that $u_m$ is a classical solution of problem (8)–(10) in the domain $Q$ and conditions (1°) and (2°) are satisfied. Then the following estimate is true for $u_m(t,x)$:*

$$|u_m(t,x)| \le \sup_k\Big(\big\|\varphi_m^{(k)};Q_k\cap(t=t_k)\big\|_0+\big\|f_m a_0^{-1};Q^{(k)}\big\|_0+\big\|g_m h_0^{-1};Q^{(k)}\big\|_0\Big). \tag{11}$$

***Proof.*** The validity of estimate (11) is established by using the same procedure as in the proof of Theorem 2.2 from [5, p. 25]. A difference is observed only in the case where $0<\max_{\bar{Q}^k} u_m = u_m(P_1)$, $P_1\in Q^{(k)}$. In view of the sufficient conditions for the existence of maximum of a function of several variables at the point $P_1$, we arrive at the following relations:

$$\partial_t u_m(P_1)\ge 0, \qquad \partial_{x_i} u_m(P_1)=0, \qquad \sum_{i,j=1}^n a_{ij}(P_1)\partial_{x_i x_j} u_m(P_1)\le 0, \tag{12}$$

and equation (8) is also satisfied.

The last inequality in (12) is true because, at the point of maximum, the second derivatives $\partial_{y_i y_j} u_m$ taken in any direction

$$y_j = \sum_{i=1}^n \alpha_{ij} s_1(\beta_i^{(1)},t^{(1)}) s_2(\beta_i^{(2)},x^{(1)})(x_i-x_i^{(1)}), \qquad \det\|\alpha_{ij}\|\ne 0,$$

are nonpositive and, moreover,

$$\sum_{i,j=1}^n a_{ij}(P_1)\partial_{x_i x_j} u_m(P_1)$$

$$= \sum_{\ell,j=1}^n \Bigg\{\sum_{i,k=1}^n s_1(\beta_i^{(1)},t^{(1)}) s_1(\beta_k^{(1)},t^{(1)}) s_2(\beta_i^{(2)},x^{(1)}) s_2(\beta_k^{(2)},x^{(1)})$$

$$\times\alpha_{k\ell}\alpha_{ij}\Bigg\}\partial_{y_j y_\ell} u_m(P_1) = \sum_{\ell=1}^n \lambda_\ell\, \partial_{y_\ell y_\ell} u_m < 0.$$

Since $\lambda_1,\ldots,\lambda_n$ are the characteristic numbers of quadratic form, they are positive according to restriction (4). In view of (12) and Eq. (8), the inequality

$$u_m(P_1) \le f_m(P_1)a_0^{-1}(P_1)$$

is true at the point $P_1$.

Similarly, if we consider the point where the function takes the least negative value for $\min\limits_{\bar{Q}^{(k)}} u_m = u_m(P_2) < 0$, $P_2 \in Q^{(k)}$, then we get

$$u_m(P_2) \ge f_m(P_2)a_0^{-1}(P_2).$$

We now estimate the derivatives of the solutions $u_m(t,x)$. In the space $C^{\ell}(Q)$, we introduce a norm $\|u_m;\gamma;\beta;q;Q\|_{\ell}$ equivalent, for fixed $m_1$ and $m_2$, to the Hölder norm defined in exactly the same way as the norm $\|u;\gamma;\beta;q;Q\|_{\ell}$ but with functions $d_1(q^{(1)},t)$ and $d_2(q^{(2)},x)$ instead of the functions $s_1(q^{(1)},t)$ and $s_2(q^{(2)},x)$, respectively:

$$d_1(q^{(1)},t) = \begin{cases} \max\{s_1(q^{(1)},t), m_1^{-q^{(1)}}\}, & q^{(1)} \ge 0, \\ \min\{s_1(q^{(1)},t), m_1^{-q^{(1)}}\}, & q^{(1)} < 0, \end{cases}$$

$$d_2(q^{(2)},x) = \begin{cases} \max\{s_2(q^{(2)},x), m_2^{-q^{(2)}}\}, & q^{(2)} \ge 0, \\ \min\{s_2(q^{(2)},x), m_2^{-q^{(2)}}\}, & q^{(2)} < 0. \end{cases}$$

The following assertion is true:

**Theorem 3.** *Assume that conditions (1°) and (2°) are satisfied. Then the solution of problem (8)–(10) satisfies the following estimate:*

$$\|u_m;\gamma,\beta,0;Q\|_{2+\alpha} \le c\sup_k\Big(\big\|\varphi_m^{(k)};\tilde{\gamma};\tilde{\beta};0;Q\cap(t=t_k)\big\|_{2+\alpha} + \|f_m;\gamma;\beta;\mu_0;Q_k\|_{\alpha}$$
$$+\big\|g_m;\gamma;\beta;\delta;Q^{(k)}\big\|_{1+\alpha} + \big\|u_m;Q^{(k)}\big\|_0\Big). \tag{13}$$

***Proof.*** Consider the problem

$$(L_1u_m)(t,x) = f_m(t,x),$$
$$(B_1u_m - g_m)(t,x)\big|_{\Gamma^{(k)}} = 0, \tag{14}$$
$$u_m(t_k+0,x) = \varphi_m^{(k)}$$

in the domains $Q^{(k)}$, $k\in\{0,1,\ldots,N\}$.

The solution of the boundary-value problem (14) in $Q^{(k)}$ exists and is unique in the space $C^{2+\alpha}(Q^{(k)})$ [5, p. 364]. To estimate this solution, we apply the interpolation inequalities from [9, 12] and obtain

$$\left\|u_m;\gamma;\beta;0;Q^{(k)}\right\|_{2+\alpha} \le (1+\varepsilon^{\alpha})\left\langle u_m;\gamma;\beta;0;Q^{(k)}\right\rangle_{2+\alpha} + c(\varepsilon)\left\|u_m;Q^{(k)}\right\|_0,$$

where $\varepsilon$ is an arbitrary real number from $(0,1)$. Therefore, it is sufficient to estimate the seminorm

$$\left\langle u_m;\gamma;\beta;0;Q^{(k)}\right\rangle_{2+\alpha}.$$

As follows from the definition of seminorm, in the domain $Q^{(k)}$, one can find points $P_1$, $P_2$, and $R_i$ for which one of the following inequalities is true:

$$\frac{1}{2}\left\|u_m;\gamma;\beta;0;Q^{(k)}\right\|_{2+\alpha} \le E_e, \qquad e\in\{1,2\}, \tag{15}$$

where

$$E_1 = \sum_{2s+|r|=2}\left\{\sum_{\nu=1}^{n} d_1(2s\gamma^{(1)},t^{(2)})d_2(2s\gamma^{(2)},\tilde{x})\prod_{i=1}^{n} d_1(r_i(\gamma^{(1)}-\beta_i^{(1)}),t^{(2)})\right.$$

$$\times d_2(r_i(\gamma^{(2)}-\beta_i^{(2)}),\tilde{x})\left|\partial_t^s\partial_x^r u_m(P_2)-\partial_t^s\partial_x^r u_m(R_\nu)\right|$$

$$\left.\times\left|x_\nu^{(1)}-x_\nu^{(2)}\right|^{-\alpha/2} d_1(\alpha(\gamma^{(1)}-\beta_\nu^{(1)}),t^{(1)})d_2(\alpha(\gamma^{(2)}-\beta_\nu^{(2)}),\tilde{x})\right\},$$

$$E_2 = d_1((2+\alpha)\gamma^{(1)},\tilde{t})d_2((2s+\alpha)\gamma^{(2)},x^{(1)})\prod_{i=1}^{n} d_1(-r_i\beta_i^{(1)},\tilde{t})$$

$$\times d_2(r_i(\gamma^{(2)}-\beta_i^{(2)}),x^{(1)})\left|t^{(1)}-t^{(2)}\right|^{-\alpha/2}$$

$$\times\left|\partial_t^s\partial_x^r u_m(P_1)-\partial_t^s\partial_x^r u_m(P_2)\right|.$$

If

$$\left|x_\nu^{(1)}-x_\nu^{(2)}\right| \ge \frac{\varepsilon_1}{4}\frac{1}{n}d_1(\gamma^{(1)},\tilde{t})d_2(\gamma^{(2)}-\beta_\nu^{(2)},\tilde{x}) \equiv T_1$$

and $\varepsilon_1$ is an arbitrary real number from $(0,1)$, then

$$E_1 \le 2\varepsilon_1^{-\alpha}\left\|u_m;\gamma;\beta;0;Q^{(k)}\right\|_2. \tag{16}$$

Moreover, if

$$\left|t^{(1)}-t^{(2)}\right| \ge \frac{\varepsilon_1^2}{16} d_1(2\gamma^{(1)},\tilde{t}) d_2(2\gamma^{(2)},\tilde{x}) \equiv T_2,$$

then

$$E_2 \le 2\varepsilon_1^{-\alpha}\left\|u_m;\gamma;\beta;0;Q^{(k)}\right\|_2. \tag{17}$$

Applying the interpolation inequalities to (16) and (17), we find

$$E_e \le \varepsilon^{\alpha}\left\|u_m;\gamma;\beta;0;Q^{(k)}\right\|_{2+\alpha} + C(\varepsilon)\left\|u_m;Q^{(k)}\right\|_0, \qquad e \in \{1,2\}. \tag{18}$$

Let $\left|x_\nu^{(1)}-x_\nu^{(2)}\right| \le T_1$ and $\left|t^{(1)}-t^{(2)}\right| \le T_2$. We assume that

$$\left|x_\nu^{(1)}-z_\nu\right| \le 4T_1, \ z \in \partial D,$$

and

$$d_1(\gamma^{(1)},\tilde{t}) = \min\{d_1(\gamma^{(1)},t^{(1)}), d_1(\gamma^{(1)},t^{(2)})\} \equiv d_1(\gamma^{(1)},t^{(1)}),$$

$$d_2(\gamma^{(2)},\tilde{x}) = \min\{d_2(\gamma^{(2)},x^{(1)}), d_2(\gamma^{(2)},x^{(2)})\} \equiv d_2(\gamma^{(2)},x^{(1)}),$$

$$P_1(t^{(1)},x^{(1)}) \in Q^{(k)}, \ k \in \{0,1,\ldots,N\}.$$

For simplicity, we set $\nu = n$. Suppose that $\mathcal{K}(P)$ is a ball of radius $R_0$, $R_0 \ge 4(T_1 n + T_2)$, centered at a point $P \in \Gamma^{(k)}$ and that the points $\{P_1, P_2, R_i\} \in \Gamma^{(k)}$. By using the restriction imposed on the smoothness of the boundary $\partial D$, we can straighten $\partial D \cap \mathcal{K}(P)$ with the help of a bijective transformation $x = \eta(\xi)$ from [12, p. 126]. As a result of this transformation, the domain $Q^{(k)} \cap \mathcal{K}(P)$ turns into the domain $D_{(k)}$ at the points of which $\xi_n \ge 0$. We assume that, under this transformation, $u_m(t,x)$, $P_1$, $P_2$, $R_i$, $E_\mu$, $d_2(\gamma^{(2)},x^{(1)})$, $T_1$, and $T_2$ are transformed into $v_m(t,\xi)$, $M_1(t^{(1)},\xi^{(1)})$, $M_2(t^{(2)},\xi^{(1)})$, $Z_i(t^{(2)},\xi^{(2)})$, $E_\mu^{(1)}$, $d_3(\gamma^{(2)},x^{(1)})$, $T_1^{(1)}$, and $T_2^{(1)}$, respectively. The coefficients of the differential expressions $L_1$ and $B_1$ in the domain $D_{(k)}$ are denoted by $\tilde{a}_{ij}(t,\xi)$, $\tilde{a}_i(t,\xi)$, $\tilde{a}_0(t,\xi)$, $\tilde{h}_k(t,\xi)$, and $\tilde{h}_0(t,\xi)$. Hence, $v_m(t,\xi)$ is a solution of the problem

$$\left[\partial_t - \sum_{i,j=1}^{n} \tilde{a}_{ij}(M_1)\partial_{\xi_i\xi_j}\right] v_m(t,\xi) = \sum_{i,j=1}^{n} [\tilde{a}_{ij}(t,\xi) - \tilde{a}_{ij}(M_1)]\partial_{\xi_i\xi_j} v_m$$

$$- \sum_{i=1}^{n} \tilde{a}_i(t,\xi)\partial_{\xi_i} v_m - \tilde{a}_0(t,\xi) v_m + f_m(t,\eta(\xi)) \equiv F_m(t,\xi;v_m), \tag{19}$$

$$v_m(t_k+0,\xi) = \varphi_m^{(k)}(\eta(\xi)), \tag{20}$$

$$\sum_{i=1}^n \tilde{h}_i(M_1)\partial_{\xi_i} v_m(t,\xi)\Big|_{\xi_n=0} = \Big\{\sum_{i=1}^n [\tilde{h}_i(M_1)-\tilde{h}_i(t,\xi)]\partial_{\xi_i} v_m - \tilde{h}_0(t,\xi)v_m$$

$$+g_m(t,\eta(\xi))\Big\}\Big|_{\xi_n=0} = G_m(t,\xi,v_m)|_{\xi_n=0}. \tag{21}$$

Let $\Pi_\tau^{(1)}$ be a domain in $D_{(k)}$:

$$\Pi_\tau^{(1)} = \{(t,\xi)\in D_{(k)} : |\xi_\nu - \xi_\nu^1| \le \tau T_1^{(1)}, \nu\in\{1,\dots,n\}, |t-t^{(1)}| \le \tau^2 T_2^{(1)}\}.$$

By the change of variables

$$v_m(t,\xi) = \omega_m(t,y), \qquad \xi_\nu = d_1(\beta_\nu^{(1)},t^{(1)})d_3(\beta_\nu^{(2)},\xi^{(1)})y_\nu,$$

in problem (19)–(21), we obtain

$$(L_2\omega_m)(t,y) \equiv \Big[\partial_t - \sum_{i,j=1}^n d_1(\beta_i^{(1)},t^{(1)})d_1(\beta_j^{(1)},t^{(1)})d_3(\beta_i^{(2)},\xi^{(1)})d_3(\beta_j^{(2)},\xi^{(1)})$$

$$\times \tilde{a}_{ij}(t^{(1)},\xi^{(1)})\partial_{y_i y_j}\Big]\omega_m = F_m(t,Y;\omega_m), \tag{22}$$

$$\omega_m(t_k+0,y) = \varphi_m^{(k)}(\eta(Y)), \tag{23}$$

$$(B_2\omega_m)(t,y)|_{y_m=0} \equiv \Big[\sum_{i=1}^n \tilde{h}_i(t^{(1)},\xi^{(1)})d_1(\beta_i^{(1)},t^{(1)})d_3(\beta_i^{(2)},\xi^{(1)})\partial_{y_i}\omega_m\Big]\Big|_{y_n=0}$$

$$= G_m(t,Y,\omega_m)|_{y_n=0}, \tag{24}$$

where

$$Y = (d_1(-\beta_1^{(1)},t^{(1)})d_3(-\beta_1^{(2)},\xi^{(1)})y_1,\dots,d_1(-\beta_n^{(1)},t^{(1)})d_3(-\beta_n^{(2)},\xi^{(1)})y_n).$$

We denote

$$y_\nu^{(1)} = d_1(-\beta_\nu^{(1)},t^{(1)})d_3(-\beta_\nu^{(2)},\xi^{(1)})x_\nu^{(1)},$$

$$V_\tau = \left\{(t,y): \left|t - t^{(1)}\right| \le \tau^2 T_2^{(1)}, \left|y_\nu - y_\nu^{(1)}\right| \le \frac{\tau}{n}\sqrt{T_2^{(1)}}\right\}$$

and choose a three times differentiable function $\psi(t,y)$ as follows:

$$\psi(t,y) = \begin{cases} 1, \quad (t,y) \in V_{1/2}, \quad 0 \le \psi(t,y) \le 1, \\ 0, \quad (t,y) \notin V_{3/4}, \quad \left|\partial_t^s \partial_x^r \psi\right| \le c_{rs} d_1(-(2s+|r|)\gamma^{(1)}, t^{(1)}) d_3(-(2s+|r|)\gamma^{(2)}, \xi^{(1)}). \end{cases}$$

Then the function $W_m(t,y) = \omega_m(t,y)\psi(t,y)$ is a solution of the boundary-value problem

$$(L_2 W_m)(t,y) \equiv \sum_{i,j=1}^{n} \tilde{a}_{ij}(t^{(1)}, \xi^{(1)}) d_1(\beta_i^{(1)}, t^{(1)}) d_1(\beta_j^{(1)}, t^{(1)}) d_3(\beta_i^{(2)}, \xi^{(1)})$$

$$\times d_3(\beta_j^{(2)}, \xi^{(1)}) \{\partial_{y_i} \omega_m \partial_{y_j} \psi + \partial_{y_j} \omega_m \partial_{y_i} \psi + \omega_m \partial_{y_i y_j} \psi\}$$

$$- \omega_m \partial_t \psi + \psi F_m \equiv F_m^{(1)}(t, Y, \omega_m), \tag{25}$$

$$W_m(t_k + 0, y) = \psi(t_k, y)\varphi_m^{(k)}(\eta(Y)) \equiv \Phi_m(t_k, y), \tag{26}$$

$$(B_2 W_m)(t,y)\Big|_{y_m=0} = \Bigg\{\psi(t,y) G_m(t, Y, \omega_m)$$

$$- \omega_m \sum_{i=1}^{n} \tilde{h}_i(t^{(1)}, \xi^{(1)}) d_1(\beta_i^{(1)}, t^{(1)})$$

$$\times d_3(\beta_i^{(2)}, \xi^{(1)}) \partial_{y_i} \psi \Bigg\}\Bigg|_{y_n=0} = \Pi_m(t, Y; \omega_m)\Big|_{y_n=0}. \tag{27}$$

The coefficients of Eq. (25) and the boundary condition (27) are limited by constants independent of the point $M_1$. Therefore, in view of Theorem 6.1 [5, p. 364], for any points $(M_3, M_4) \subset V_{1/2}$, the following inequality is true:

$$d^{-\alpha}(M_3, M_4)\left|\partial_t^s \partial_y^r \omega_m(M_3) - \partial_t^s \partial_y^r \omega_m(M_4)\right|$$

$$\le c\left(\left\|F_m^{(1)}\right\|_{C^\alpha(V_{3/4})} + \left\|\Phi_m\right\|_{C^{2+\alpha}(V_{3/4} \cap (t=t_k))} + \left\|\Pi_m\right\|_{C^{1+\alpha}(V_{3/4})}\right), \qquad 2s + |r| = 2, \tag{28}$$

where $d(M_3, M_4)$ is the parabolic distance between the points $M_3$ and $M_4$.

In view of the properties of the function $\psi(t,y)$, we estimate the norms of the expressions $F_m^{(1)}$, $\Phi_m$, and $\Pi_m$ as follows:

$$\left\|F_m^{(1)}\right\|_{C^{\alpha}(V_{3/4})} \le cd_1(-(2+\alpha)\gamma^{(1)},t^{(1)})d_3(-(2+\alpha)\gamma^{(2)},\xi^{(1)})$$

$$\times(\left\|f_m;\gamma;0,2\gamma;V_{3/4}\right\|_{\alpha}+\left\|\omega_m;V_{3/4}\right\|_0+\left\|\omega_m;\gamma;0,0;V_{3/4}\right\|_2), \quad (29)$$

$$\left\|\Phi_m\right\|_{C^{2+\alpha}(V_{3/4}\cap(t=t_k))} \le cd_1(-(2+\alpha)\gamma^{(1)},t^{(1)})d_3(-(2+\alpha)\gamma^{(2)},\xi^{(1)})$$

$$\times\left\|\varphi_m^{(k)};\tilde{\gamma};0;0;V_{3/4}\cap(t=t_k)\right\|_{2+\alpha}, \quad (30)$$

$$\left\|\Pi_m\right\|_{C^{1+\alpha}(V_{3/4})} \le cd_1(-(2+\alpha)\gamma^{(1)},t^{(1)})d_3(-(2+\alpha)\gamma^{(2)},\xi^{(1)})$$

$$\times(\left\|G_m;\gamma;0;\gamma;V_{3/4}\right\|_{\alpha}+\left\|\omega_m;V_{3/4}\right\|_0+\left\|\omega_m;\gamma;0,0;V_{3/4}\right\|_2). \quad (31)$$

It follows from the definition of $H^{\ell}(\gamma;\beta;q;Q)$ that the following inequalities are true:

$$c_1\left\|\omega_m;\gamma;0;0;V_{3/4}\right\|_{\ell} \le \left\|v_m;\gamma;\beta;0;\Pi_{3/4}^{(1)}\right\|_{\ell} \le c_2\left\|\omega_m;\gamma;0;0;V_{3/4}\right\|_{\ell}.$$

Substituting (29)–(31) in (28) and returning to the variables $(t,x)$, we arrive at inequalities

$$E_{\mu} \le (\varepsilon^{\alpha}(n+2)+\varepsilon_1 Cn^2)\left\|u_m;\gamma;\beta;0;Q^{(k)}\right\|_{2+\alpha}+C\Big(\left\|u_m;Q^{(k)}\right\|_0$$

$$+\left\|f_m;\gamma;\beta;2\gamma;Q^{(k)}\right\|_{\alpha}+\left\|g_m;\gamma;\beta;\gamma;Q^{(k)}\right\|_{1+\alpha}$$

$$+\left\|\varphi_m^{(k)};\tilde{\gamma};\tilde{\beta};0;Q^{(k)}\cap(t=t_k)\right\|_{2+\alpha}\Big). \quad (32)$$

Suppose that $\left|x_{\nu}^{(1)}-x_{\nu}^{(2)}\right|\le T_1$, $\left|t^{(1)}-t^{(2)}\right|\le T_2$, and $\left|x_{\nu}^{(1)}-z_{\nu}\right|\ge 4T_1$, $z\in\partial D$. In the domain $Q^{(k)}$ we represent problem (14) as follows:

$$\left[\partial_t-\sum_{i,j=1}^{n}a_{ij}(P_1)\partial_{x_i x_j}\right]u_m = \sum_{i,j=1}^{n}[a_{ij}(P)-a_{ij}(P_1)]\partial_{x_i x_j}u_m-\sum_{i=1}^{n}a_i(P)\partial_{x_i}u_m$$

$$-a_0(P)u_m+f_m(t,\lambda) \equiv F(t,x,u_m),$$

$$u_m(t_k+0,x) = \varphi_m^{(k)}(x),$$

$$\sum_{i=1}^{n} h_i(P_1)\partial_{x_i} u_m \Big|_{\Gamma^{(k)}} = \left\{ \sum_{i=1}^{n} [h_i(P_1) - h_i(P)]\partial_{x_i} u_m - h_0(P)u_m + g_m(P) \right\}\Bigg|_{\Gamma^{(k)}}$$

$$= \Phi_m(t, x, u_m)\big|_{\Gamma^{(k)}}.$$

Repeating the reasoning presented above and using Theorem 5.3 from [5, p. 364], we obtain the inequality

$$E_\mu \le (\varepsilon^\alpha (n+2) + \varepsilon_1 C n^2)\left\|u_m;\gamma;\beta;0;Q^{(k)}\right\|_{2+\alpha} + C\Big(\left\|u_m;Q^{(k)}\right\|_0$$

$$+\left\|f_m;\gamma;\beta;2\gamma;Q^{(k)}\right\|_\alpha + \left\|g_m;\gamma;\beta;\gamma;Q^{(k)}\right\|_{1+\alpha}$$

$$+\left\|\varphi_m^{(k)};\tilde{\gamma};\tilde{\beta};0;Q^{(k)}\cap(t=t_k)\right\|_{2+\alpha}\Big). \tag{33}$$

Combining inequalities (15), (18), (32), and (33), we get estimate (13).

***Proof of Theorem 1.*** Since

$$\left\|f_m;\gamma;\beta;\mu_0;Q^{(k)}\right\|_\alpha \le c\left\|f;\gamma;\beta;\mu_0;Q^{(k)}\right\|_\alpha,$$

$$\left\|g_m;\gamma;\beta;\delta;Q^{(k)}\right\|_{1+\alpha} \le c\left\|g;\gamma;\beta;\delta;Q^{(k)}\right\|_{1+\alpha},$$

$$\left\|\varphi_m^{(k)};\tilde{\gamma};\tilde{\beta};0;Q\cap(t=t_k)\right\|_{2+\alpha} \le c\left\|\varphi_k;\tilde{\gamma};\tilde{\beta};0;Q\cap(t=t_k)\right\|_{2+\alpha},$$

by virtue of inequalities (11) and (13), we find

$$\|u_m;\gamma;\beta;Q\|_{2+\alpha} \le c\sup_k\Big(\left\|\varphi_k;\tilde{\gamma};\tilde{\beta};0;Q\cap(t=t_k)\right\|_{2+\alpha} + \left\|f;\gamma;\beta;\mu_0;Q^{(k)}\right\|_\alpha$$

$$+\left\|g;\gamma;\beta;\delta;Q^{(k)}\right\|_{1+\alpha}\Big). \tag{34}$$

The right-hand side of inequality (34) is independent of $m_1$ and $m_2$, and the sequences

$$\{U_m^{(0)} \equiv u_m(P)\},$$

$$\{U_m^{(1)} \equiv d_1(\gamma^{(1)} - \beta_i^{(1)}, t)d_2(\gamma^{(2)} - \beta_i^{(2)}, x)\partial_{x_i} u_m\},$$

$$\{U_m^{(2)} \equiv d_1(2\gamma^{(1)}, t)d_2(2\gamma^{(2)}, x)\partial_t u_m\},$$

$$\{U_m^{(3)} \equiv d_1(\gamma^{(1)} - \beta_i^{(1)}, t) d_1(\gamma^{(1)} - \beta_j^{(1)}, t) d_2(\gamma^{(2)} - \beta_i^{(2)}, x)$$

$$\times d_2(\gamma^{(2)} - \beta_j^{(2)}, x) \partial_{x_i x_j} u_m\}, \qquad P(t,x) \in Q^{(k)},$$

are uniformly bounded and equicontinuous in $Q^{(k)}$. According to the Arzelà theorem, there exist subsequences $\{U_{m(\ell)}^{(\mu)}\}$ uniformly convergent to $U^{\mu}$ in $Q^{(k)}$, $\mu \in \{0,1,2,3\}$. Passing to the limit as $m(\ell) \to \infty$ in problem (8)–(10), we conclude that $u(t,x) = U_0^{(0)}$ is the unique solution of problem (1)–(3), $u \in H^{2+\alpha}(\gamma;\beta;0;Q)$, and estimate (5) is true.

Since $H^{\ell}(\gamma;\beta;q;Q) \subset H^{\ell}(\gamma;\beta;0;Q)$, for $f(t,x) \in H^{\alpha}(\gamma;\beta;0;Q)$ and $g(t,x) \in H^{1+\alpha}(\gamma;\beta;0;Q)$ we get the following estimates:

$$\left\| f;\gamma;\beta;\mu_0;Q^{(k)} \right\|_{\alpha} \le c \left\| f;\gamma;\beta;0;Q^{(k)} \right\|_{\alpha},$$

$$\left\| g;\gamma;\beta;\delta;Q^{(k)} \right\|_{1+\alpha} \le c \left\| g;\gamma;\beta;0;Q^{(k)} \right\|_{1+\alpha}. \tag{35}$$

By using estimates (5) and (35) for the solution of problem (1)–(3), we establish the validity of the estimate

$$\|u;\gamma;\beta;0;Q\|_{2+\alpha} \le c \sup_k \Big( \left\| \varphi_k;\tilde{\gamma};\tilde{\beta};0;Q \cap (t=t_k) \right\|_{2+\alpha} + \left\| f;\gamma;\beta;0;Q^{(k)} \right\|_{\alpha}$$

$$+ \left\| g;\gamma;\beta;0;Q^{(k)} \right\|_{1+\alpha} \Big). \tag{36}$$

Note that the space

$$H_{\alpha} = H^{\alpha}(\gamma;\beta;0;Q^{(k)}) \times H^{2+\alpha}(\tilde{\gamma};\tilde{\beta};0;Q \cap (t=t_k))$$

is embedded in the space $C(Q^{(k)})$, $H_{\alpha} \subset C(Q^{(k)})$. Therefore, by the Riesz theorem, we can assume that, for fixed $(t,x) \in Q^{(k)}$, the linear continuous functional $u(t,x)$ generates a Borel measure $G(t,x,Z^{(k)})$ defined on a $\sigma$-algebra of the subsets $Z^{(k)}$ of the domain $Q^{(k)}$, including $Q^{(k)}$ and all its open subsets for which the value of the functional is given by formula (6).

As follows from Theorem 2, inequalities (7) are satisfied for the solution of problem (1)–(3):

$$|u_1| \equiv \left| \int_{Q_k} G_1^{(k)}(t,x;d\tau,d\xi) f(\tau,\xi) \right| \le \left\| f A_0^{-1};Q^{(k)} \right\|_0,$$

$$|u_2| \equiv \left| \int_D G_2^{(k)}(t,x;0,d\xi) \varphi_k(\xi) \right| \le \|\varphi_k;Q \cap (t=t_k)\|_0, \tag{37}$$

$$|u_3| \equiv \left| \int_{\Gamma^{(k)}} G_3^{(k)}(t,x;d\tau,d_{\xi}S) g(\tau,\xi) \right| \le \left\| g b_0^{-1};Q^{(k)} \right\|_0,$$

where $u_1$ is an arbitrary solution of problem (1)–(3) for $g \equiv 0$ and $\varphi_k \equiv 0$; $u_2$ is a solution of the boundary-value problem (1)–(3) for $f \equiv 0$ and $g \equiv 0$, and $u_3$ is a solution of boundary-value problem (1)–(3) for $f \equiv 0$ and $\varphi_k \equiv 0$.

Substituting $f(t,x) \equiv 1$, $\varphi_k(x) \equiv 1$, and $g(t,x) \equiv 0$ in inequalities (37), we arrive at inequalities (6).

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