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Research Article

Pseudodifferential Equation of Fluctuations of Nonstationary Gravitational Fields

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Developing Holtzmark's idea, the distribution of nonstationary fluctuations of local interaction of moving objects of the system with gravitational influence, which is characterized by the Riesz potential, is constructed. A pseudodifferential equation with the Riesz fractional differentiation operator is found, which corresponds to this process. The general nature of symmetric stable random Lévy processes is determined.

1. Introduction

One of the main problems of celestial mechanics is the analysis of the nature of the force of interaction between objects in one or another star system. Force $\mathcal{F}$ acting on a specific star of the system has two components: the first one, K, is the influence of the entire system and the second, F, is the local influence of the immediate environment: $\mathcal{F} = K + F$.

The influence of the entire system can be described by the gravitational potential $\mathcal{R}(r;t)$ [1] the average spatial distribution of stars of different masses m at time t. Such force, K, related to the unit of mass acting on the star Z_0 under consideration by the system (as a whole), is determined by the equality:

$$K(r;t) = -\mathrm{grad}\mathcal{R}(r;t). \tag{1}$$

The force $K(r;t)$ is a function with a slow change in space and time since the corresponding potential $\mathcal{R}(r;t)$ characterizes the "smoothed" distribution of matter in the stellar system, whereas the other force $F(t)$, related to the unit of mass, has relatively fast, sharp changes caused by instantaneous changes in the local distribution of stars from the environment Z_0 at time t. The value of $F(t)$ is subject to fluctuations, so we can only talk about its probabilistic value.

The famous Danish astronomer Holtzmark studied the stochastic properties of $F(t)$ [1, 2]. He used Newton's gravitational law, according to which

$$F(t) = \sum_{j=1}^{N(t)} F_j = G\sum_{j=1}^{N(t)} \frac{m_j}{|r_j|^2} r_j^{\circ}, \tag{2}$$

where G is the gravitational constant, m_j is the mass of a typical star Z_j, r_j is the radius-vector of Z_j, $r_j^{\circ} := r_j/|r_j|$, $|r_j|^2 = (r_j, r_j)$, scalar square in the Euclidean space $\mathbb{R}^3$, and $N(t)$ is the number of stars that at the moment t form the local environment Z_0. He assumed the constancy of the average density $n(r;m;t) \equiv n$ of the spatial distribution of stars as well as the fulfillment of the equality:

$$N = \frac{4}{3}\pi R^3 n, \quad (\forall R > 0). \tag{3}$$

Using classical means of probability theory in combination with integral calculus, Holtzmark found a stationary distribution $W(F)$ of the quantity F in the form

$$W(F) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{-i(\xi,F) - a|\xi|^{3/2}} d\xi \equiv \mathbb{F}^{-1}\left[e^{-a|\xi|^{3/2}}\right](F), \tag{4}$$

where $a := (4/15)(2\pi G)^{3/2} n\langle m^{3/2}\rangle$ is the fluctuation coefficient, in which $\langle m^{3/2}\rangle$ is the average value of $m^{3/2}$, which corresponds to the observed law of stars distribution in the stellar system, and $\mathbb{F}$ is the Fourier transform operator.

This Holtzmark distribution belongs to the class of distributions

$$\mathscr{L}_\lambda^\alpha(x) = \mathbb{F}^{-1}\left[e^{-\lambda|\xi|^\alpha}\right](x), \quad \lambda > 0, x \in \mathbb{R}^3, \tag{5}$$

of symmetric stable random processes described by the French researcher Zolotarev et al. [3, 4] at the beginning of the last century.

The appearance of the class of distributions $\mathscr{L}_\lambda^\alpha(\cdot)$, in addition to these Holtzmark studies, was preceded by the studies of Poisson and Cauchy. About a hundred years before the publication of [4], first Poisson and then Cauchy paid attention to the distribution with density:

$$P_\lambda(x) = \frac{\lambda}{\pi\left(\lambda^2 + x^2\right)}, \quad \lambda > 0, x \in \mathbb{R}, \tag{6}$$

which has the property

$$P_{\lambda_1}(\cdot) * P_{\lambda_2}(\cdot) = P_\lambda(\cdot), \tag{7}$$

and the parameter λ is uniquely determined by the values λ_1 and λ_2. Within his statistical studies, being interested in the cases, when the error of an individual observation might be comparable to the mean error in a series of large numbers of independent observations, Cauchy found that such a case would occur if the observation errors were subject to a probability distribution with a density $P_\lambda(\cdot)$. It should be noted that

$$P_\lambda(\cdot) = \mathscr{L}_\lambda^1(\cdot). \tag{8}$$

Cauchy [5] knew that each of the functions $\mathscr{L}_\lambda^\alpha(\cdot)$ for $\alpha > 0$ possesses property (7), but he could not select the nonnegative functions from them (which were, obviously, the densities of some probability distributions) except for (8). The famous Hungarian mathematician Polya [6] managed to prove that $\mathscr{L}_\lambda^\alpha \geq 0$ for $\alpha \in (0; 1)$ only at the beginning of the last century. The fact that the functions $\mathscr{L}_\lambda^\alpha(\cdot)$ are densities of probability distributions only for $\alpha \in (0; 2]$ was fully determined by Le vy in 1924 [4].

The Gauss–Wiener distribution of

$$\mathscr{L}_t^2(x) = \frac{1}{\left(2\sqrt{t\pi}\right)^3} e^{-\left(|x|^2/4t\right)}, \quad t > 0, x \in \mathbb{R}^3. \tag{9}$$

Brownian motion in the space $\mathbb{R}^3$, according to which heat and diffusion propagate in 3D environment, is also a good representative of the class $\mathscr{L}_\lambda^\alpha(\cdot)$. In the scientific literature [7–12], there are many examples of real applications of the Holtzmark, Cauchy, and Gauss distributions in astronomy, nuclear physics, economics, in the industrial and military sectors, etc. Each of these applications characterizes the stochastic features of Le vy distributions with one or another value of α.

However, the symmetric stable random processes $\mathscr{L}_\lambda^\alpha(\cdot)$, besides their individual characteristics, have a general nature. In this paper, it is determined that each of such Lévy processes at $\alpha \in (0; 2)$ can be regarded as a process of local influence of moving objects in the gravitational field of M. Rees, i.e., in a system, in which the interaction between masses occurs according to a certain power law $(\cdot)^{-\beta}$. In particular, the classical Holzmark process ($\alpha = 3/2$) corresponds to the interaction with $\beta = 2$ (the case of Newtonian gravity), while the Cauchy process corresponds to the interaction with $\beta = 3$. Here, we also consider the Holtzmark problem in a general formulation and obtain a pseudo-differential equation (PDE) with the Riesz operator of fractional differentiation, which corresponds to the Holtzmark random nonstationary processes. The presence of this equation allows one to study Holtzmark stochastic processes in domains with boundary by means of the theory of boundary value problems for the PDE.

The main content of the work is as follows. In Section 2, the stochastic problem of the local influence of moving objects in the evolutionary gravitational field is studies. The connection of the Holtzmark fluctuations' distributions of nonstationary gravitational fields with one classical PDE is determined in Section 3. A brief historical overview of this equation study and its further application is also presented here. Section 4 is the conclusions.

2. The Problem of the Local Influence of Stars

Let us consider a star system, in which the interaction between masses is subordinate to M. Riesz potential [13]. It means that the gravity between two arbitrary stars of the corresponding masses M and m is described by the law:

$$F = G\frac{Mm}{|r|^\beta} r^\circ, \quad \beta > 0, \tag{10}$$

where G is a certain gravity constant and r is a vector of distance between these stars. Let the star Z_0 under consideration be at the origin of the coordinate system and $F(t)$ be the force acting on the unit of mass of the star Z_0 from its closest environment at time t.

Developing Holtsmark's idea, we will find the nonstationary distribution $W_\beta(F(t))$ for the force $F(t)$. For this, we also assume that the distribution of stars in the neighborhood Z_0 is subject to fluctuations according to some empirically established law and that, at each moment of time t, the fluctuations of the density of stars are subject to the condition of the constancy of their average density per unit volume:

$$n(r; m; t) \equiv n(t). \tag{11}$$

If we now assume that the star Z_0 under study is at the origin of the system and its spherical circle of radius R at time t contains $N(t)$ stars, then, according to the above,

$$F(t) = G\sum_{j=1}^{N(t)} \frac{m_j}{\left|r_j\right|^{\beta+1}} r_j \equiv \sum_{j=1}^{N(t)} F_j, \tag{12}$$

$$N(t) = \frac{4}{3}\pi R^3 n(t). \tag{13}$$

First, let us fix t and consider the distribution of $W_{\beta,N(t)}(F(t))$ in the center of the spherical vicinity of the radius R, which contains $N(t)$ of stars of the system. Let us find the probability $W_{\beta,N(t)}(F_\circ(t))dF_\circ(t)$ that the quantity $F(t)$ will be cubed $[_{F^\circ}(t); F_\circ(t)+dF_\circ(t)] \subset \mathbb{R}^3$. Using the well-known method of characteristic functions, we obtain

$$W_{\beta,N(t)}\left(F_\circ(t)\right) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{-i(\xi,F_\circ(t))} A_{N(t)}(\xi)\mathrm{d}\xi, \quad (14)$$

where

$$A_{N(t)}(\xi) := \prod_{j=1}^{N(t)} \int_0^{+\infty}\left(\int_{\mathbb{K}_R(0)} e^{i(\xi,F_j)}\tau_j\left(r_j;m_j;t\right)\mathrm{d}r_j\right)\mathrm{d}m_j, \quad (15)$$

Here, $\mathbb{K}_R(0)$ is a sphere of radius R with the center at the origin and $\tau_j(r_j;m_j;t)$ is the distribution of probability that, at time t, the star Z_j has mass m_j and is in position r_j.

Since it is believed that there are only fluctuations of stars compatible with the constancy of their spatial average density, then

$$\tau_j\left(r_j;m_j;t\right) = \frac{3\tau(m;t)}{4\pi R^3}, \quad (16)$$

where $\tau(m;t)$ is the frequency, with which stars of different masses meet at time t.

Taking it into consideration, we will receive the equality:

$$A_{N(t)}(\xi) = \left(\frac{3}{4\pi R^3}\int_0^{+\infty}\left(\int_{\mathbb{K}_R(0)} e^{i(\xi,\eta)}\tau(m;t)\mathrm{d}r\right)\mathrm{d}m\right)^{N(t)}, \quad (17)$$

in which

$$\eta := \frac{Gmr}{|r|^{\beta+1}}. \quad (18)$$

Directing $R \longrightarrow +\infty$ and $N(t) \longrightarrow +\infty$, according to condition (13), we obtain

$$W_\beta(F(t)) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{-i(\xi,F(t))}A(\xi;t)\mathrm{d}\xi, \quad (19)$$

where

$$A(\xi;t) := \lim_{R\longrightarrow\infty}\left[\frac{3}{4\pi R^3}\int_0^{+\infty}\left(\int_{\mathbb{K}_R(0)} e^{i(\xi,\eta)}\tau(m;t)\mathrm{d}r\right)\mathrm{d}m\right]^{4\pi R^3 n(t)/3}. \quad (20)$$

For all values of t, the equality

$$\frac{3}{4\pi R^3}\int_0^{+\infty}\left(\int_{\mathbb{K}_R(0)} \tau(m;t)\mathrm{d}r\right)\mathrm{d}m = 1, \quad (21)$$

is true, so

$$A(\xi;t) = \lim_{R\longrightarrow\infty}\left[1-\frac{3}{4\pi R^3}\int_0^{+\infty}\left(\int_{\mathbb{K}_R(0)}\left(1-e^{i(\xi,\eta)}\right)\tau(m;t)\mathrm{d}r\right)\mathrm{d}m\right]^{4\pi R^3 n(t)/3}. \quad (22)$$

Furthermore, the absolute convergence in (22) of the integral with the integration variable r in the entire space $\mathbb{R}^3$ at $\beta > (3/2)$ allows us to write equation (22) in the form

$$A(\xi;t) = \lim_{R\longrightarrow\infty}\left[1-\frac{3}{4\pi R^3}\int_0^{+\infty}\left(\int_{\mathbb{R}^3}\left(1-e^{i(\xi,\eta)}\right)\tau(m;t)\mathrm{d}r\right)\mathrm{d}m\right]^{4\pi R^3 n(t)/3}. \quad (23)$$

From here, we get the image

$$A(\xi;t) = e^{-n(t)B_\beta(\xi;t)}, \quad (24)$$

in which

$$B_\beta(\xi;t) := \int_0^{+\infty}\left(\int_{\mathbb{R}^3}\left(1-e^{i(\xi,\eta)}\right)\tau(m;t)\mathrm{d}r\right)\mathrm{d}m. \quad (25)$$

The last equality can be written in the form

$$B_\beta(\xi;t) = \frac{4\pi(G|\xi|)^{3/\beta} < m^{3/\beta} >}{\beta}\int_0^{+\infty}(\rho-\sin\rho)\rho^{-2-3/\beta}\mathrm{d}\rho. \quad (26)$$

For this, it is necessary to replace the integration variable r with the variable η in the inner integral of this equality according to rule (18) and then to turn to a spherical coordinate system, in which the z-axis is directed along the ξ vector.

It should be noted that the integral of the last equation coincides only at $\beta > 3/2$. Integrating it by parts, we arrive at the equality

$$B_\beta(\xi;t) = \frac{4\beta\pi I(\beta)}{3(\beta+3)}(G|\xi|)^{3/\beta}\langle m^{3/\beta}\rangle, \quad t \geq 0, \xi \in \mathbb{R}^3, \beta > \frac{3}{2}, \tag{27}$$

where

$$I(\beta) := \begin{cases} \frac{\beta}{3-\beta}\Gamma\left(\frac{2-3}{\beta}\right)\cos\frac{(2-3/\beta)\pi}{2}, & \frac{3}{2} < \beta < 3, \\ \frac{\pi}{2}, & \beta = 3, \\ \Gamma\left(\frac{1-3}{\beta}\right)\sin\frac{(1-3/\beta)\pi}{2}, & \beta > 3, \end{cases} \tag{28}$$

where $\Gamma(\cdot)$ is Euler's gamma function.

Combining equalities (19), (24), and (27), we finally find

$$W_\beta(F(t)) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{-i(\xi,F(t))}e^{-a_\beta(t)|\xi|^{3/\beta}}\mathrm{d}\xi, \tag{29}$$

where

$$a_\beta(t) := \frac{4\beta\pi I(\beta)}{3(\beta+3)}G^{3/\beta}n(t)\langle m^{3/\beta}\rangle. \tag{30}$$

Thus, we have proved the following theorem.

Theorem 1. *With the above assumptions, for every $\beta > 3/2$, the function*

$$W_\beta(F(t)) = \mathbb{F}^{-1}\left[e^{-a_\beta(t)|\xi|^{3/\beta}}\right](F;t), \tag{31}$$

is the probabilities distribution of the force $F(t)$ of the local influence of moving objects in the system with the interaction that takes place according to the power law (10).

Remark 1. For $\beta \leq 3/2$, the integral $I(\beta)$ diverges, so the coefficient $a_\beta(\cdot)$ becomes immeasurable. Therefore, we conclude that there is no corresponding random Holtzmark process in the sense of the problem of local interaction of moving objects.

Let us introduce the notation $\mathscr{H}_\gamma(F;t) := W_\beta(F(t))$, where $\gamma := 2/\beta$. The function $\mathscr{H}_\gamma(\cdot;\cdot)$ is called the Holtzmark distribution of the order γ of fluctuations of nonstationary gravitational fields. The classical situation considered by Holtzmark corresponds to $\beta = 2$; in this case, the distribution order $\gamma = 1$. Taking the ratio $\beta > 3/2$ into account, this is the only case of the integer order γ, and the rest of the possible values of γ have a nonzero fractional part: $\gamma \in (0;4/3)$.

In Section 3, we find the corresponding differential equation, the fundamental solution of which is the Holtzmark distribution $\mathscr{H}_\gamma(\cdot;\cdot)$.

3. Connection with the PDE

It would be more desirable to study the fluctuations in the local interaction of moving objects, especially in a limited environment with certain conditions at the boundary, by reducing to solve the corresponding boundary value problems for differential or pseudodifferential equations. This would allow us to use the advanced computing apparatus of the theory of boundary value problems and use its known results. In this regard, there is a need to obtain an appropriate differential equation that adequately reflects the process under study. Under certain conditions, we will try to derive this equation "starting" from the distribution function $\mathscr{H}_\gamma$. To do this, let us first find out the properties of this function.

It will be assumed here that the coefficient $a_\beta(\cdot)$ is a positive, continuous-differential function on the interval $[0;T]$. It follows directly from [14, 15] that, for all $\gamma \in (0;4/3)$, the function $\mathscr{H}_\gamma(x;t)$ on the set $\mathbb{R}^n \times (0;T]$ is differentiable by t and infinitely differentiable by the variable x; for its derivatives, the following estimates are fulfilled:

$$\begin{aligned} \left|\partial_x^k \mathscr{H}_\gamma(x;t)\right| &\leq c_1 t\left(t^{(2/3\gamma)} + |x|\right)^{-(3+|k|+(3\gamma/2))}, \\ \left|\partial_t\partial_x^k \mathscr{H}_\gamma(x;t)\right| &\leq c_2 t^{(2/\gamma)-1}\left(t^{(2/3\gamma)} + |x|\right)^{-(3+|k|+(3\gamma/2))}, \end{aligned} \tag{32}$$

with some positive constants c_1 and c_2.

Estimate (32) ensures that $\mathscr{H}_\gamma(\cdot;t)$ belongs to the Lebesgue class $L_1(\mathbb{R}^3)$ for each fixed $t \in (0;T]$. Therefore, this guarantees the existence of the Fourier transform of the function $\mathscr{H}_\gamma(\cdot;t)$ and satisfies the equality

$$\mathbb{F}\left[\mathscr{H}_\gamma(x;t)\right](\xi;t) = e^{-a_\beta(t)|\xi|^{(3\gamma/2)}}, \quad t \in (0;T], \xi \in \mathbb{R}^3. \tag{33}$$

We shall arbitrarily fix $t \in [0;T]$, and for $\triangle t \neq 0$, we consider

$$\mathscr{H}_\gamma(x;t+\triangle t) - \mathscr{H}_\gamma(x;t) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{-i(x,\xi)-a_\beta(t)|\xi|^{(3\gamma/2)}}\left(e^{-\left(a_\beta(t+\triangle t)-a_\beta(t)\right)|\xi|^{(3\gamma/2)}} - 1\right)\mathrm{d}\xi. \tag{34}$$

According to Lagrange's theorem on finite increments, we have

$$a_\beta(t+\triangle t) - a_\beta(t) = a_\beta'(t+\theta\triangle t)\triangle t, \quad \theta \in (0;1). \tag{35}$$

Hence, given the continuity of $a_\beta'(\cdot)$, we obtain that

$$\frac{\left(e^{-\left(a_\beta(t+\triangle t)-a_\beta(t)\right)|\xi|^{(3\gamma/2)}} - 1\right)}{\triangle t} \overset{\xi \in \mathbb{K}_R(0)}{\rightrightarrows} -a_\beta'(t)|\xi|^{(3\gamma/2)}, \quad (\forall R > 0). \tag{36}$$

In addition, using Lagrange's theorem again, we find

$$\left|\frac{\left(e^{-\left(a_\beta'(t+\theta\triangle t)|\xi|^{(3\gamma/2)}\right)\triangle t}-1\right)}{\triangle t}\right| = \left|a_\beta'(t+\theta\triangle t)\right||\xi|^{(3\gamma/2)}e^{-a_\beta\prime(t+\theta\triangle t)|\xi|^{(3\gamma/2)}(\widehat{\theta}\triangle t)} \leq$$
$$\leq a|\xi|^{(3\gamma/2)}e^{a|\triangle t||\xi|^{(3\gamma/2)}},\quad \widehat{\theta}\in(0;1),$$
$$a := \sup_{t\in[0;T]}\left|a_\beta'(t)\right|. \tag{37}$$

Then, for all $0<|\triangle t|\leq a_\beta(t)/(2a)$ and $\xi\in\mathbb{R}^3$, the following estimates are true:

$$e^{-a_\beta(t)|\xi|^{(3\gamma/2)}}\left|\frac{\left(e^{-\left(a_\beta(t+\triangle t)-a_\beta(t)\right)|\xi|^{(3\gamma/2)}}-1\right)}{\triangle t}\right|$$
$$\leq a|\xi|^{(3\gamma/2)}e^{-\left(a_\beta(t)-a|\triangle t|\right)|\xi|^{(3\gamma/2)}}\leq$$
$$\leq 4ae^{-\left(a_\beta(t)/4\right)|\xi|^{(3\gamma/2)}}\sup_{\rho>0}\{\rho e^{-\rho}\}. \tag{38}$$

Relations (36) and (38) ensure the correctness of the equality

$$\lim_{\triangle t\longrightarrow 0}\int_{\mathbb{R}^3}e^{-i(x,\xi)-a_\beta(t)|\xi|^{(3\gamma/2)}}\cdot\frac{e^{-\left(a_\beta(t+\triangle t)-a_\beta(t)\right)|\xi|^{(3\gamma/2)}}-1}{\triangle t}d\xi =$$
$$=\int_{\mathbb{R}^3}e^{-i(x,\xi)-a_\beta(t)|\xi|^{(3\gamma/2)}}\cdot\lim_{\triangle t\longrightarrow 0}\left\{\frac{e^{-\left(a_\beta(t+\triangle t)-a_\beta(t)\right)|\xi|^{(3\gamma/2)}}-1}{\triangle t}\right\}d\xi, \tag{39}$$

according to which, from (34), we obtain

$$\partial_t\mathcal{H}_\gamma(x;t) = -\frac{a_\beta'(t)}{(2\pi)^3}\int_{\mathbb{R}^3}|\xi|^{(3\gamma/2)}$$
$$e^{-i(x,\xi)-a_\beta(t)|\xi|^{(3\gamma/2)}}d\xi,\quad t\in(0;T],x\in\mathbb{R}^3. \tag{40}$$

Now, taking (33) into account, we finally arrive at

$$\partial_t\mathcal{H}_\gamma(x;t) = -a_\beta'(t)\mathbb{F}^{-1}\left[|\xi|^{(3\gamma/2)}\mathbb{F}\left[\mathcal{H}_\gamma\right](\xi;t)\right](x;t),\quad t\in(0;T],x\in\mathbb{R}^3. \tag{41}$$

Thus, the Holtzmark distribution H_γ is a classical solution of the equation

$$\partial_t u(x;t) + a_\beta'(t)A_\nu u(x;t) = 0,\quad t\in(0;T],x\in\mathbb{R}^3, \tag{42}$$

with Riesz operator A_ν fractional differentiation of the $\nu := (3\gamma/2)$ order [16, 17].

Furthermore, we clear up the question of existence of the limit value of distribution $\mathcal{H}_\gamma(\cdot;t)$ at the point $t=0$.

First, let us consider the case $a_\beta(0)\neq 0$. According to equation (33) and the well-known Fourier transform formula for convolution of elements of the Lebesgue class $L_1(\mathbb{R}^3)$, we obtain that

$$\mathcal{H}_\gamma(x;t) = \left(G_\nu * \widehat{\mathcal{H}}_\gamma\right)(x;t),\quad t\in(0;T],\xi\in\mathbb{R}^3, \tag{43}$$

where $\widehat{\mathcal{H}}_\gamma(\cdot) := \mathcal{H}_\gamma(\cdot;0)$ is the corresponding Holtzmark stationary distribution, and

$$G_\nu(\cdot;t) := \mathbb{F}^{-1}\left[e^{-\int_0^t a_\beta'(\tau)d\tau|\xi|^\nu}\right](\cdot;t). \tag{44}$$

We now show that, for every continuous function $\mathbb{R}^3$ limited by $\varphi(\cdot)$, the boundary relation holds:

$$\left(G_\nu * \varphi\right)(\cdot;t)\underset{t\longrightarrow +0}{\longrightarrow}\varphi(\cdot). \tag{45}$$

For that, we shall use the equality

$$\int_{\mathbb{R}^3}G_\nu(x;t)dx = 1,\quad t\in(0;T], \tag{46}$$

according to which

$$\left|\left(G_\nu * \varphi\right)(x;t)-\varphi(x)\right|$$
$$\leq\int_{\mathbb{R}^3}\left|G_\nu(\xi;t)\right||\varphi(x-\xi)-\varphi(x)|d\xi\equiv\mathfrak{I}(x;t). \tag{47}$$

Since $\varphi(\cdot)$ is a continuous function on $\mathbb{R}^3$, then, for every $x\in\mathbb{R}^3$ and arbitrary $\varepsilon>0$, there exists $t_\circ$ such that $t_\circ^{(1/2\nu)}<\varepsilon$ and $|\varphi(x-\xi)-\varphi(x)|<\varepsilon$ if $|\xi|<t_\circ^{(1/2\nu)}$. Then,

$$\mathfrak{I}(x;t)<\varepsilon\int_{|\xi|<t_\circ^{(1/2\nu)}}\left|G_\nu(\xi;t)\right|d\xi+\int_{|\xi|\geq t_\circ^{(1/2\nu)}}$$
$$\left|G_\nu(\xi;t)\right||\varphi(x-\xi)-\varphi(x)|d\xi\leq\varepsilon\mathfrak{I}_1(t)+\mathfrak{I}_2(x;t), \tag{48}$$

where

$$\mathfrak{I}_1(t) := \int_{\mathbb{R}^3}\left|G_\nu(\xi;t)\right|d\xi,$$
$$\mathfrak{I}_2(x;t) := \int_{|\xi|\geq t_\circ^{(1/2\nu)}}\left|G_\nu(\xi;t)\right||\varphi(x-\xi)-\varphi(x)|d\xi. \tag{49}$$

Furthermore, having considered the estimates [14, 15]

$$\left|\partial_x^k G_\nu(x;t)\right|\leq c_1 t\left(t^{1/\nu}+|x|\right)^{-(3+|k|+\nu)},\quad k\in\mathbb{Z}_+^3, t\in(0;T],x\in\mathbb{R}^3, \tag{50}$$

and the limited function $\varphi(\cdot)$ in $\mathbb{R}^3$, for all $t \in (0;T]$ and $x \in \mathbb{R}^3$, we find

$$\mathfrak{J}_1(t) \le c_1 t \int_{\mathbb{R}^3} \frac{\mathrm{d}\xi}{\left(t^{1/\nu} + |\xi|\right)^{3+\nu}} = c_1 \int_{\mathbb{R}^3} \frac{\mathrm{d}z}{(1+|z|)^{3+\nu}} \equiv c_2,$$

$$\mathfrak{J}_2(x;t) \le c_3 t \int_{|\xi| \ge t_\circ^{(1/2\nu)}} |\xi|^{-(3+\nu)} \mathrm{d}\xi = c_3 t \int_{t_\circ^{(1/2\nu)}}^{+\infty} \rho^{-(1+\nu)} \mathrm{d}\rho = c_4 t_{t_\circ}^{-1/2}. \tag{51}$$

It follows from the last inequality that, for all $x \in \mathbb{R}^3$ and $t \le t_\circ$,

$$\mathfrak{J}_2(x;t) \le c_{4t_\circ}^{1/2} < c_4 \varepsilon^\nu. \tag{52}$$

Consequently,

$$\mathfrak{J}(x;t) < c_2 \varepsilon + c_4 \varepsilon^\nu, \quad \forall x \in \mathbb{R}^3 \forall \varepsilon > 0 \exists t_\circ < \varepsilon^{2\nu} \forall t \le t_\circ, \tag{53}$$

i.e., the boundary relation (45) is satisfied.

The function $\widehat{\mathscr{H}}_\gamma(\cdot)$ is infinitely differentiable and is limited to $\mathbb{R}^3$, so, according to (45), for $\mathscr{H}_\gamma(\cdot;t)$, the relation is fulfilled:

$$\mathscr{H}_\gamma(\cdot;t) \underset{t \longrightarrow +0}{\longrightarrow} \widehat{\mathscr{H}}_\gamma(\cdot). \tag{54}$$

Thus, the distribution $\mathscr{H}_\gamma(x;t)$ is a classical solution of the Cauchy problems (42) and (54).

Now, let $a_\beta(0) = 0$; then, the next equality follows directly from (33):

$$\mathscr{H}_\gamma(\cdot;t) = G_\nu(\cdot;t), \quad t \in (0;T]. \tag{55}$$

It should be noted that relation (45) characterizes the property "δ-similarity" of the function $G_\nu(\cdot;t)$ in the space S' of Schwartz distributions [18]:

$$G_\nu(\cdot;t) \underset{t \longrightarrow +0}{\longrightarrow} \delta(\cdot), \tag{56}$$

where $\delta(\cdot)$ is the Dirac delta function. Therefore, for $a_\beta(0) = 0$, the Holtzmark distribution $\mathscr{H}_\gamma(\cdot;t)$ is the solution of the Cauchy problems (42) and (56), which in the usual sense satisfies equation (42), and it satisfies the initial condition (56) in the sense of weak convergence in the space S'. This solution G_ν is called Green's function of the Cauchy problem for equation (42).

Let us summarize the above in the form of the following statement.

Theorem 2. *Let $\beta > 3/2$ and $a_\beta(\cdot)$ be positive, continuous-differential functions on the interval $(0;T]$; then, for $a_\beta(0) \ne 0$, the corresponding Holtzmark distribution $\mathscr{H}_{2/\beta}(\cdot;t)$ on the set $\mathbb{R}^3 \times (0;T]$ is a classical solution of the Cauchy problems (42 and 54). If $a_\beta(0) = 0$, then $\mathscr{H}_{2/\beta}(\cdot;t)$ is Green's function of this problem.*

Remark 2. Equation (55) reveals the meaning of Green's function of the Cauchy problem for equation (42): G_ν is the primary Holtzmark distribution of the local influence on the object under consideration by its moving environment characterizing this process from the very beginning of its origin, i.e., from the moment when the elements of local influence first appeared in the environment of the object.

The study of Green's function of the Cauchy problem for the PDE of form (42) was initiated by S.D. Eidelman and Ya.M. Drin in the early 80s of last century [19, 20]. They proposed a method for constructing and studying the function G_ν based on the Fourier transform and obtained the following estimates:

$$\left|\partial_x^k G_\nu(x;t)\right| \le c_1 t \left(t^{1/\nu} + |x|\right)^{-(n+|k|+[\nu])}, \quad k \in \mathbb{Z}_+^n, t \in (0;T], x \in \mathbb{R}^n, \tag{57}$$

where $[\cdot]$ is the integer part of a number. However, this method imposes restrictions on the order of ν PDE: $\nu > 1$.

The accurate asymptotic behavior of Green's function $G_\nu(\cdot;t)$ in the vicinity of infinitely distant points was determined by Fedoryuk in [21]:

$$G_\nu(\cdot;t) \sim |\cdot|^{-n-\nu}, \quad t > 0. \tag{58}$$

Subsequently, Schneider [22], effectively using the Mellin transform, expresses the function $G_\nu(\cdot;t)$ through special Fox's H functions and, as a consequence, obtains asymptotics (58). It should be noted that long before the publication of [21], the asymptotics (58) for the PDE (42) at $a_\beta(t) = t$ and $\nu \in [0;1]$ was described by R.M. Blumenthal and R.K. Getoor in [23].

A new approach to the study of the properties of the function $G_\nu(\cdot;t)$, which is based on the use of the elements of the theory of generalized functions and harmonic analysis, was applied by Kochubei in [24]. He was the first to receive estimates (57), in which $[\nu]$ is replaced by ν, in the case when the dimension of the spatial variable is greater than unity and $\nu \ge 1$.

In the works of [14, 15], developing the idea of [24], Kochubei's estimates are extended in the case of $\nu > 0$.

Remark 3. In a partial case $a_\beta(t) = c_0 t + c_1, c_0 > 0, c_1 \ge 0$, equation (42) is known as the equation of random walk with arbitrarily long jumps. It has important applications in modern ecology, biology, economics, and physics and is also closely related to Le vy's random flights (see, for example, [25]).

Pseudodifferential equations of type (42) and their generalizations as well as the application of these equations in the theory of stable random processes have been considered by many researchers [25–36].

4. Conclusions

According to Theorem 1, each Holzmark distribution $\mathscr{H}_\gamma$ is a Le vy distribution $\mathscr{L}_\lambda^\alpha$ of the order $\alpha = 3\gamma/2$. Since $\gamma \in (0;4/3)$, then the γ-spectrum of Holtzmark distributions coincides with the α-spectrum of Le vy distributions with the accuracy up to one nearest boundary value $\alpha = 2$, which corresponds to the diffusion process. This value $\alpha = 2$ corresponds to $\gamma = 4/3$, which is also the boundary value of the γ-Holzmark spectrum. However, for such γ, there is no random Holzmark process. This means that the process of classical diffusion occurs according to the laws that differ in nature from the laws of random vortices of local gravity of moving objects.

Taking into account the results of the research in Section 3, equation (42) can be called the pseudodifferential equation of local fluctuation of nonstationary Riesz gravitational fields.

Data Availability

The data used to support the findings of the study are available at https://doi.org/10.1007/s11253-006-0040-6 and https://doi.org/10.1007/s11202-008-0030-z.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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