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NONLOCAL (IN TIME) PROBLEM FOR THE EVOLUTIONARY EQUATION WITH FRACTIONAL DIFFERENTIAL OPERATOR

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We establish the correct solvability of a multipoint nonlocal (in time) problem for the evolutionary equation with operator of fractional differentiation and an initial function, which is an element of the space of generalized functions of the distribution type. The analytic representation of the solution is presented. We also analyze the behavior of the solution in the case of unlimited growth of the time variable (stabilization of the solution).

Various classical function spaces (e.g., Sobolev spaces, spaces of analytic functions, spaces of infinitely differential functions, and spaces of Schwartz distributions) can be understood as positive and negative spaces with respect to L_2 constructed either on the basis of functions of the operator of differentiation or of multiplication by an independent variable or as projective and inductive limits of these spaces [1]. In the cited work, the nonlocal multipoint (in time) problem was investigated in the half space $t > 0$ for the differential-operator equation

$$\frac{\partial u}{\partial t} + Au = 0,$$

where $A = |D_x|^\alpha$, $D_x = d/dx$, and $\alpha \in (1, +\infty)\setminus\{2, 3, 4, \ldots\}$ is the fractional power of modulus of the operator of differentiation. This operator can be regarded as an analog of the Weyl operator of fractional differentiation, which is used in the theory of periodic functions [2, 3]. The analyzed problem is a generalization of the Cauchy problem in the case where the initial condition $u(t, \cdot)|_{t=0} = f$ is replaced by the condition

$$\sum_{k=0}^{m} \mu_k u(t, \cdot)|_{t=t_k} = f,$$

where $t_0 = 0$, $\{t_1, \ldots, t_m\} \subset (0, +\infty)$, $0 < t_1 < t_2 < \ldots < t_m < +\infty$, and $\{\mu_0, \mu_1, \ldots, \mu_m\} \subset \mathbb{R}$, $m \in \mathbb{N}$, are fixed numbers (for $\mu_0 = 1$, $\mu_1 = \ldots = \mu_m = 0$, we get the Cauchy problem). This condition is treated either in the classic sense or in the weak sense if f is a generalized function, i.e., in a sense of the limit relation

$$\sum_{k=0}^{m} \mu_k \lim_{t \to t_k} \langle u(t, \cdot), \varphi \rangle = \langle f, \varphi \rangle,$$

for any function φ from the pivot space (here, $\langle f, \varphi \rangle$ denotes the action of a functional f on a test function). This problem is nonlocal in time and belongs to the class of multipoint problems for differential-operator equations

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(for a survey of works devoted to nonlocal problems for differential-operator equations and partial differential equations, see, e.g., [4]).

In the present paper, we establish correct solvability of the indicated problem with initial function, which is an element of the space of generalized functions of the distribution type, and present the analytic representation of the solution $u(t,x)$ as $t \to +\infty$ (stabilization of the solution). It is shown that every pseudodifferential operator (in the case of one independent variable) constructed according to a homogeneous function of order α, which is not differentiable at the point 0, coincides with the restriction of the operator A to a certain locally convex topological space, which is a projective limit of Banach spaces continuously embedded into each other.

1. Spaces of Test and Generalized Functions

1.1. The Space Φ_α. Let α be a fixed number from the set $(1,+\infty)\setminus\{2,3,4,\ldots\}$, let $\alpha_0 := [\alpha]+1$, ($[\alpha]$ is the integral part of a number α), let $M(x) := 1+|x|$, $x\in\mathbb{R}$, and let

$$\Phi_\alpha := \left\{\varphi\in C^\infty(\mathbb{R}) \;\middle|\; \forall k\in\mathbb{Z}_+ \;\; \exists c_k = c_k(\varphi)>0 \;\; \forall x\in\mathbb{R}\colon M^{\alpha_0+k}(x)|\varphi^{(k)}(x)|\le c_k\right\}.$$

In Φ_α, we introduce the structure of countably normed space with the help of norms

$$\|\varphi\|_p := \sup_{x\in\mathbb{R}}\left\{\sum_{k=0}^{p} M^{\alpha_0+k}(x)|\varphi^{(k)}(x)|\right\}, \quad \varphi\in\Phi_\alpha, \quad p\in\mathbb{Z}_+.$$

Moreover (see [5, pp. 103–110]), Φ_α is a complete perfect countably normed space with the topology of projective limit of Banach spaces

$$\Phi_{p,\alpha}:\; \Phi_\alpha = \lim_{p\to\infty}\operatorname{pr}\Phi_{p,\alpha} = \bigcap_{p=0}^{\infty}\Phi_{p,\alpha}$$

($\Phi_{p,\alpha}$ is the complement of Φ_α with respect to the pth norm) and the embeddings $\Phi_{p+1,\alpha}\subset\Phi_{p,\alpha}$, $p\in\mathbb{Z}_+$, are continuous.

The function

$$\varphi(x) = \left(1+x^2\right)^{-([\alpha]+1)/2}, \quad x\in\mathbb{R}, \quad \alpha\in(1,+\infty)\setminus\{2,3,\ldots\}$$

can be regarded as an example of an element of the space Φ_α.

By induction, we can show that the derivatives of this function have the form

$$\varphi^{(k)}(x) = P_k(x)\left(1+x^2\right)^{-(k+([\alpha]+1)/2)}, \quad x\in\mathbb{R}, \quad k\in\mathbb{Z}_+,$$

where P_k is a polynomial of degree k. This implies that all norms $\|\varphi\|_p$, $p\in\mathbb{Z}_+$, are finite.

A set $B\subset\Phi_\alpha$ is called *bounded* if

$$\forall p\in\mathbb{Z}_+ \quad \exists c_p>0 \quad \forall\varphi\in B\colon \|\varphi\|_p\le c_p$$

(i.e., each norm of the space Φ_α is bounded on the set B by its constant).

A sequence of functions $\{\varphi_\nu, \nu \geq 1\} \subset \Phi_\alpha$ converges in Φ_α to a function $\varphi \in \Phi_\alpha$ as $\nu \to +\infty$ if $\|\varphi_\nu - \varphi\|_p \to 0$ as $\nu \to +\infty$ for any $p \in \mathbb{Z}_+$.

Note that a continuous operation of shift of the argument $T_\xi: \varphi(x) \to \varphi(x+\xi)$, $\varphi \in \Phi_\alpha$ and the operation of differentiation are also defined in the space Φ_α (see [5, pp. 108–110]). Since Φ_α is a perfect space, according to the general results of the theory of perfect spaces [6, pp. 171–172], the operation of shift of the argument in the space Φ_α is not only continuous but also infinitely differentiable (i.e., limit relations of the form $(\varphi(x+h)-\varphi(x))h^{-1} \to \varphi'(x)$, $h \to 0$, are true in a sense of convergence in the topology of the space Φ_α).

1.2. The Space Ψ_α. Since functions from the space Φ_α are absolutely integrable on $\mathbb{R}$, the operation of Fourier transformation F can be defined for these functions as follows:

$$F[\varphi](\sigma) = \int_{\mathbb{R}} \varphi(x) e^{i\sigma x}\, dx, \quad \varphi \in \Phi_\alpha.$$

By Ψ_α we denote the Fourier transform of the space Φ_α: $\Psi_\alpha = F[\Phi_\alpha]$. It is clear that each function $F[\varphi]$, $\varphi \in \Phi_\alpha$, is bounded and continuous on $\mathbb{R}$. We now consider main properties of functions from the space Ψ_α (see [7, pp. 197–210]):

(i) If $\varphi \in \Phi_\alpha$, then $F[\varphi] \in L_1(\mathbb{R})$.

(ii) If $\varphi \in \Phi_\alpha$, then $F[\varphi]$ is an infinitely differentiable function on $\mathbb{R} \setminus \{0\}$.

Note that the function $F[\varphi]$ can be not differentiable at the point $\sigma = 0$. Thus, the function $\varphi(x) = (1+x^2)^{-1}$, $x \in \mathbb{R}$, is an element of the space Φ_α, $\alpha \in (1, 2)$. However, it is known that

$$F[\varphi](\sigma) = \pi \exp\{-|\sigma|\}, \qquad \sigma \in \mathbb{R}.$$

This function is not differentiable at the point $\sigma = 0$. Another example: the function $\varphi(x) = (1+x^2)^{-m}$, $x \in \mathbb{R}$, $m \in \mathbb{N}$, is an element of the space Φ_α, $\alpha \in (2m-1, 2m)$. In this case (see [8, pp. 364–367]),

$$F[\varphi](\sigma) = \frac{2\pi}{(m-1)!} \sum_{k=0}^{m-1} \frac{(m+k-1)!}{k!(m-k-1)!2^{m+k}} |\sigma|^{m-k-1} e^{-|\sigma|}, \quad \sigma \in \mathbb{R}.$$

These examples characterize elements from Φ_α as Fourier preimages of functions nonsmooth at the point 0.

(iii) The function $D_\sigma^k F[\varphi](\sigma)$, $\varphi \in \Phi_\alpha$ $\sigma \neq 0$, $k \in \mathbb{N}$, has finite unilateral limits $\lim_{\sigma \to \pm 0} D_\sigma^k F[\varphi](\sigma)$.

(iv) The Fourier transformation continuously and bijectively maps Φ_α onto Ψ_α.

(v) Functions from the space Ψ_α satisfy the condition

$$\forall k \in \mathbb{Z}_+ \quad \exists c_k = c_k(\psi) > 0\colon \sup_{\sigma \in \mathbb{R}\setminus\{0\}} \left|\sigma^k \psi^k(\sigma)\right| \leq c_k, \quad \psi \in \Psi_\alpha.$$

In Ψ_α, we introduce the structure of countably normed space with the help of the norms

$$\|\psi\|_p := \sup_{\sigma \in \mathbb{R}\setminus\{0\}} \left\{ \sum_{k=0}^{p} \left|\sigma^k \psi^{(k)}(\sigma)\right| \right\}, \quad \psi \in \Psi_\alpha, \quad p \in \mathbb{Z}_+.$$

In this case, Ψ_α is a complete countably normed space, $\Psi_\alpha = \bigcap_{p=0}^{\infty} \Psi_{p,\alpha}$, where $\Psi_{p,\alpha}$ is the complement of the space Ψ_α with respect to the norm $\|\cdot\|_p$ and the embedding $\Psi_{p+1,\alpha} \subset \Psi_{p,\alpha}$, $p \in \mathbb{Z}_+$, is continuous.

A function $g \in C(\mathbb{R}) \cap C^\infty(\mathbb{R} \setminus \{0\})$ (or $g \in C^\infty(\mathbb{R})$) is called a *multiplicator* in the space Ψ_α if $g\psi \in \Psi_\alpha$ for any function $\psi \in \Psi_\alpha$ and the mapping $\psi \to g\psi$ is a linear and continuous operator from Ψ_α into Ψ_α.

1.3. The Space Φ'_α. By Φ'_α we denote the space of all linear continuous functionals defined on Φ_α with weak convergence. Since the topology of projective limit of the Banach spaces Φ_α is introduced in the pivot space Φ_α and the embeddings $\Phi_{p+1,\alpha} \subset \Phi_{p,\alpha}$, $p \in \mathbb{Z}_+$, are continuous, we conclude that [1, pp. 53–54]

$$\Phi'_\alpha = \left(\lim_{p\to\infty} \operatorname{pr} \Phi_{p,\alpha} \right)' = \lim_{p\to\infty} \operatorname{ind} \Phi'_{p,\alpha}.$$

Thus, if $f \in \Phi'_\alpha$, then $f \in \Phi'_{p,\alpha}$ for some $p \in \mathbb{Z}_+$. The least of these p is called the order of f and, moreover,

$$|\langle f, \varphi \rangle| \le c\|\varphi\|_p, \quad \varphi \in \Phi_\alpha,$$

where $c = \|f\|_p$ is the norm of the functional f in the space $\Phi'_{p,\alpha}$. The space Φ'_α is complete.

1.4. Convolution in Φ'_α. If $f \in \Phi'_\alpha$, $\varphi \in \Phi_\alpha$, then the convolution $f * \varphi$ exists and is given by the formula (see [5, pp. 112–118])

$$(f * \varphi)(x) := \left\langle f_\xi, T_{-x}\check{\varphi}(\xi) \right\rangle, \quad \check{\varphi}(\xi) := \varphi(-\xi);$$

moreover, $f * \varphi$ is an ordinary function infinitely differentiable on $\mathbb{R}$ (here, $\langle f_\xi, T_{-x}\check{\varphi}(\xi)\rangle$ denotes the action of a functional f upon a test function $T_{-x}\check{\varphi}(\xi)$ regarded as a function of the argument ξ).

Let $f \in \Phi'_\alpha$. If $f * \varphi \in \Phi_\alpha$ for any function $\varphi \in \Phi_\alpha$ and the fact that $\varphi_\nu \to 0$ as $\nu \to +\infty$ in the space Φ_α implies that $f * \varphi_\nu \to 0$ as $\nu \to +\infty$ in the space Φ_α, then the functional f is called a *convolver* in the space Φ_α.

The Fourier transform of a generalized function $f \in \Phi'_\alpha$ is defined by the formula

$$\langle F[f], F[\varphi] \rangle = 2\pi \langle f, \varphi \rangle, \qquad \varphi \in \Phi_\alpha.$$

This implies that $F[f] \in \Psi'_\alpha$ for $f \in \Phi'_\alpha$. If f is a convolver in the space Φ_α, then

$$F[f * \varphi] = F[f] * F[\varphi] \quad \forall \varphi \in \Phi_\alpha.$$

In this case, $F[f]$ is a multiplicator in the space Ψ_α (see [7, p. 219]).

For example, the Dirac δ-function (δ: $\langle \delta, \varphi \rangle := \varphi(0)$) is a convolver in the space Φ_α, i.e.,

$$\delta * \varphi = \langle \delta_\xi, T_{-x}\check{\varphi}(\xi) \rangle = T_{-x}\check{\varphi}(0) = \check{\varphi}(-x) = \varphi(x) \quad \forall \varphi \in \Phi_\alpha.$$

At the same time, the function $F[\delta] = 1$ is a multiplicator in the space Ψ_α.

2. Fractional Differentiation in the Space Φ_α

Consider an operator

$$A = |D_x|^\alpha = |iD_x|^\alpha, \qquad D_x = d/dx,$$

which is a fractional power of the modulus of the operator of differentiation. The operator A is a nonnegative self-adjoint operator in the Hilbert space $L_2(\mathbb{R})$ because iD_x is a self-adjoint operator in $L_2(\mathbb{R})$ with the domain of definition

$$\mathcal{D}(iD_x) = \{\varphi \in L_2(\mathbb{R}) \mid \exists \varphi' \in L_2(\mathbb{R})\}.$$

If E_λ, $\lambda \in \mathbb{R}$, is the spectral function of the operator iD_x, then, in view of the main spectral theorem for self-adjoint operators, we get

$$A\varphi = |D_x|^\alpha \varphi = \int_{-\infty}^{+\infty} |\lambda|^\alpha \, dE_\lambda \varphi,$$

$$\varphi \in \mathcal{D}(A) = \left\{ \varphi \in L_2(\mathbb{R}) \colon \int_{-\infty}^{+\infty} |\lambda|^{2\alpha} \, d(E_\lambda \varphi, \varphi) < \infty \right\}.$$

In view of the form of the spectral function E_λ of the operator iD_x (see [9, p. 421]), we conclude that an arbitrary function $\varphi \in \Phi_\alpha \subset L_2(\mathbb{R})$ satisfies the relation

$$E_\lambda \varphi(t) = \frac{1}{2\pi} \int_{-\infty}^{\lambda} \left\{ \int_{-\infty}^{+\infty} \varphi(\tau) e^{-i\sigma\tau} d\tau \right\} e^{-it\sigma} d\sigma = \frac{1}{2\pi} \int_{-\infty}^{\lambda} F[\varphi](\sigma) e^{-it\sigma} \, d\sigma.$$

This yields

$$dE_\lambda \varphi = \frac{1}{2\pi} F[\varphi](\lambda) e^{-it\lambda} d\lambda.$$

Hence,

$$A\varphi = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\lambda|^\alpha F[\varphi](\lambda) e^{-it\lambda} \, d\lambda = F^{-1}\big[|\lambda|^\alpha F[\varphi]\big], \quad \varphi \in \Phi_\alpha. \tag{1}$$

To substantiate relation (1), we prove the following statement:

Lemma 1. *The function* $\chi(\sigma) = |\sigma|^\alpha$, $\sigma \in \mathbb{R}$, *is a multiplicator in the space* Ψ_α.

Proof. We take an arbitrary function $\psi \in \Psi_\alpha$ and prove that $\chi \cdot \psi \in \Psi_\alpha$. To this end, it suffices to show that, for $\sigma \neq 0$,

$$\forall p \in \mathbb{Z}_+ \quad \exists c_p > 0 \colon \sum_{k=0}^{p} |\sigma|^k \left| (\chi(\sigma)\psi(\sigma))^{(k)} \right| \leq c_p.$$

Let $\sigma\colon |\sigma| \geq 1$. Then

$$|\sigma|^k \left|(\chi(\sigma)\psi(\sigma))^{(k)}\right| = |\sigma|^k \sum_{l=0}^{k} C_k^l \left|\left(|\sigma|^\alpha\right)^{(l)}\right| \left|\psi^{(k-l)}(\sigma)\right|$$

$$\leq |\sigma|^k \sum_{l=0}^{k} C_k^l |\alpha(\alpha-1)\dots(\alpha-(l-1))||\sigma|^{\alpha-l} \left|\psi^{(k-l)}(\sigma)\right|$$

$$= \sum_{l=0}^{k} C_k^l a_l |\sigma|^\alpha |\sigma|^{k-l} \left|\psi^{(k-l)}(\sigma)\right|, \quad a_l = |\alpha(\alpha-1)\dots(\alpha-(l-1))|.$$

Further, we use the following inequality from [7, p. 206]:

$$\left|\sigma^k \psi^{(k)}(\sigma)\right| \leq \frac{b_k}{|\sigma|^{1+[\alpha]}}, \quad |\sigma| \geq 1, \quad k \in \mathbb{Z}_+,$$

which is true for any function $\psi \in \Psi_\alpha$. Thus, for $\sigma\colon |\sigma| \geq 1$, we obtain

$$|\sigma|^k \left|(\chi(\sigma)\psi(\sigma))^{(k)}\right| \leq \sum_{l=0}^{k} C_k^l a_l b_{k-l} \frac{|\sigma|^\alpha}{|\sigma|^{1+[\alpha]}} \leq \sum_{l=0}^{k} C_k^l a_l b_{k-l} \equiv \tilde{c}_k.$$

Hence, for $\sigma\colon |\sigma| \geq 1$, we can write

$$\sum_{k=0}^{p} |\sigma|^k \left|(\chi(\sigma)\psi(\sigma))^{(k)}\right| \leq \sum_{k=0}^{p} \tilde{c}_k \equiv \tilde{c}_p.$$

If $\sigma \neq 0\colon |\sigma| < 1$, then

$$|\sigma|^k \left|(\chi(\sigma)\psi(\sigma))^{(k)}\right| \leq \sum_{l=0}^{k} C_k^l a_l |\sigma|^{k-l} \left|\psi^{(k-l)}(\sigma)\right|.$$

By using the inequality $|\sigma|^k |\psi^{(k)}(\sigma)| \leq c_k$, $\sigma \neq 0$, $k \in \mathbb{Z}_+$ (see Property 5 in Sec. 1), we arrive at the inequality

$$|\sigma|^k \left|(\chi(\sigma)\psi(\sigma))^{(k)}\right| \leq \sum_{l=0}^{k} C_k^l a_l c_{k-l} \equiv \tilde{\tilde{c}}_p, \quad \sigma \neq 0, \quad |\sigma| < 1.$$

Thus,

$$\forall p \in \mathbb{Z}_+ \quad \exists c_p > 0\colon \|\chi\psi\|_p \leq c_p,$$

i.e., $\chi\psi \in \Psi_\alpha$. The operator $\Psi_\alpha \ni \psi \to \chi\psi \in \Psi_\alpha$ is bounded because it maps every bounded set from the space Ψ_α into a bounded set from the same space (this property is proved by using the scheme described above). Note that, in the space Ψ_α, as a space with the first axiom of countability, the class of linear bounded operators coincides with the class of linear continuous operators. This implies that the function $\chi(\sigma)$, $\sigma \in \mathbb{R}$, is a multiplicator in the space Ψ_α.

Let $\hat{A} := A|_{\Phi_\alpha}$ be the restriction of the operator A to Φ_α. By Lemma 1, the operator $\hat{A}$ maps Φ_α into Φ_α. Moreover, it is a linear operator and coincides on Φ_α with a pseudodifferential operator $F^{-1}[\chi(\sigma)F]$ constructed on the basis of the function symbol $\chi(\sigma) = |\sigma|^\alpha$, $\sigma \in \mathbb{R}$.

Remark 1. If $a(\sigma)$, $\sigma \in \mathbb{R}$, is a homogeneous function of order $\alpha \in (1, +\infty) \setminus \{2, 3, 4, \ldots\}$ continuous on $\mathbb{R}$ and infinitely differentiable on $\mathbb{R} \setminus \{0\}$, then it is known that $a(\sigma)$ has the form $a(\sigma) = c|\sigma|^\alpha$, $c = \text{const}$. According to the reasoning presented above, in the case of one independent variable, every pseudodifferential operator constructed on the basis of the function $a(\sigma)$, $\sigma \in \mathbb{R}$, coincides with the operator $\hat{A} = A|_{\Phi_\alpha}$.

3. Nonlocal (in Time) Problem

Consider an evolutionary equation

$$\frac{\partial u(t,x)}{\partial t} + \hat{A}u(t,x) = 0, \quad (t,x) \in (0,+\infty) \times \mathbb{R} \equiv \Omega, \tag{2}$$

where $\hat{A}$ is the operator of fractional differentiation in the space Φ_α considered in Sec. 2.

A solution of Eq. (2) is understood as a function $u(t,x)$, $(t,x) \in \Omega$ with the following properties:

(i) it is continuously differentiable with respect to the variable t;

(ii) $u(t,\cdot) \in \Phi_\alpha$ for any $t > 0$;

(iii) $u(t,x)$, $(t,x) \in \Omega$, satisfies Eq. (2).

For Eq. (2), we now formulate the nonlocal (in time) multipoint problem as follows:

To find the solution of Eq. (2) satisfying the condition

$$\mu u(0,x) - \mu_1 u(t_1,x) - \ldots - \mu_m u(t_m,x) = f(x), \quad x \in \mathbb{R}, \quad f \in \Phi_\alpha, \tag{3}$$

where $u(0,x) = \lim\limits_{t \to +0} u(t,x)$, $x \in \mathbb{R}$, $m \in \mathbb{N}$, $\{\mu, \mu_1, \ldots, \mu_m\} \subset (0,+\infty)$, $\{t_1, \ldots, t_m\} \subset (0,+\infty)$ are fixed numbers, $0 < t_1 < \ldots < t_m < +\infty$, and

$$\mu > \sum_{k=1}^{m} \mu_k.$$

We seek the solution of problem (2), (3) by using the Fourier transform in the form

$$u(t,x) = F^{-1}[v(t,\sigma)].$$

For the function $v \colon \Omega \to \mathbb{R}$, we obtain the following problem with a parameter σ:

$$\frac{dv(t,\sigma)}{dt} + |\sigma|^\alpha v(t,\sigma) = 0, \quad (t,\sigma) \in \Omega, \tag{4}$$

$$\mu v(0,\sigma) - \sum_{k=1}^{m} \mu_k v(t_k,\sigma) = \tilde{f}(\sigma), \quad \sigma \in \mathbb{R}, \tag{5}$$

where $\tilde{f}(\sigma) = F[f](\sigma)$. A solution of problem (4), (5) is given by the formula

$$v(t,\sigma) = \tilde{f}(\sigma) \exp\{-t|\sigma|^{\alpha}\} \left(\mu - \sum_{k=1}^{m} \mu_k \exp\{-t_k|\sigma|^{\alpha}\}\right)^{-1}, \quad \sigma \in \mathbb{R}.$$

Thus, the function

$$u(t,x) = (2\pi)^{-1} \int_{\mathbb{R}} v(t,\sigma) e^{-ix\sigma} d\sigma$$

is the solution of problem (2), (3).

We introduce the notation $G(t,x) = F^{-1}[Q(t,\sigma)]$, where

$$Q(t,\sigma) = Q_1(t,\sigma) Q_2(\sigma), \qquad Q_1(t,\sigma) = \exp\{-t|\sigma|^{\alpha}\},$$

$$Q_2(\sigma) = \left(\mu - \sum_{k=1}^{m} \mu_k \exp\{-t_k|\sigma|^{\alpha}\}\right)^{-1}.$$

Reasoning formally, we get

$$u(t,x) = \int_{\mathbb{R}} G(t,x-\xi) f(\xi) d\xi = G(t,x) * f(x).$$

Indeed,

$$u(t,x) = (2\pi)^{-1} \int_{\mathbb{R}} Q(t,\sigma) \left(\int_{\mathbb{R}} f(\xi) e^{i\sigma\xi} d\xi \right) e^{-i\sigma x}\, d\sigma$$

$$= \int_{\mathbb{R}} \left((2\pi)^{-1} \int_{\mathbb{R}} Q(t,\sigma) e^{-i\sigma(x-\xi)}\, d\sigma \right) f(\xi)\, d\xi$$

$$= \int_{\mathbb{R}} G(t,x-\xi) f(\xi)\, d\xi = G(t,x) * f(x), \quad (t,x) \in \Omega.$$

The validity of transformations performed above follows from the properties of the function $Q(t,\sigma)$ regarded as a function of the variable σ, which are presented in what follows.

Lemma 2. *For fixed $t > 0$, the function $Q(t,\sigma)$ is infinitely differentiable with respect to the variable $\sigma \in \mathbb{R} \setminus \{0\}$. Its derivatives satisfy the estimates*

$$|D_\sigma^s Q(t,\sigma)| \le b_s t^{\gamma s} |\sigma|^{\omega_s - s} \exp\{-t|\sigma|^\alpha\}, \quad \sigma \ne 0, \quad s \in \mathbb{N}, \tag{6}$$

where the constant $b_s = b_s(\alpha) > 0$ *is independent of* t,

$$\gamma = \begin{cases} 0 & \text{for} \quad 0 < t \le 1, \\ 1 & \text{for} \quad t > 1, \end{cases}$$

$$\omega_s = \begin{cases} \alpha & \text{for} \quad \sigma \ne 0, \quad |\sigma| < 1, \\ \alpha s & \text{for} \quad |\sigma| \ge 1. \end{cases}$$

Proof. To prove the lemma, we use the Faa di Bruno formula of differentiation of a composite function:

$$D_\sigma^s F(g(\sigma)) = \sum_{\tilde m=1}^{s} \frac{d^{\tilde m} F(g)}{dg^{\tilde m}} \sum \frac{s!}{\tilde m_1! \dots \tilde m_l!}$$

$$\times \left(\frac{d}{d\sigma} g(\sigma)\right)^{\tilde m_1} \left(\frac{1}{2!} \frac{d^2}{d\sigma^2} g(\sigma)\right)^{\tilde m_2} \dots \left(\frac{1}{l!} \frac{d^l}{d\sigma^l} g(\sigma)\right)^{\tilde m_l}, \tag{7}$$

where the index of summation runs over all solutions of the equation in nonnegative integers,

$$\tilde m_1 + 2\tilde m_2 + \dots + l\tilde m_l = s, \quad \tilde m_1 + \tilde m_2 + \dots + \tilde m_l = \tilde m.$$

In this formula, we set $F = e^g$, $g = -t|\sigma|^\alpha$. Then

$$|D_\sigma^s \exp\{-t|\sigma|^\alpha\}| \le e^{-t|\sigma|^\alpha} \sum_{\tilde m=1}^{s} \sum \frac{s!}{\tilde m_1! \dots \tilde m_l!} \Lambda, \quad \sigma \ne 0, \quad s \in \mathbb{N},$$

where

$$\Lambda := \left| \left(\frac{d}{d\sigma} \left(-t|\sigma|^\alpha\right)\right)^{\tilde m_1} \left(\frac{1}{2!} \frac{d^2}{d\sigma^2} g(\sigma)\right)^{\tilde m_2} \dots \left(\frac{1}{l!} \frac{d^l}{d\sigma^l} g(\sigma)\right)^{\tilde m_l} \right|, \quad \sigma \ne 0.$$

Since $\alpha > 1$, the inequalities

$$\alpha(\alpha-1)(\alpha-2)\dots(\alpha-(l-1)) \le \alpha(\alpha+1)\dots(\alpha+l) \le \alpha \cdot 2\alpha \cdot 3\alpha \dots l\alpha = \alpha^l l!$$

are true. By using the last inequality, we arrive at the following estimate for Λ:

$$\Lambda \le t^{\tilde m_1} \alpha^{\tilde m_1} |\sigma|^{(\alpha-1)\tilde m_1} t^{\tilde m_2} \alpha^{2\tilde m_2} |\sigma|^{(\alpha-2)\tilde m_2} \dots t^{\tilde m_l} \alpha^{l\tilde m_l} |\sigma|^{(\alpha-l)\tilde m_l}$$

$$= t^{\tilde m_1 + \dots + \tilde m_l} \alpha^{\tilde m_1 + 2\tilde m_2 + \dots + l\tilde m_l} |\sigma|^{\alpha(\tilde m_1 + \dots + \tilde m_l) - (\tilde m_1 + 2\tilde m_2 + \dots + l\tilde m_l)}$$

$$= t^{\tilde{m}} \alpha^s |\sigma|^{\alpha\tilde{m}-s}, \quad \sigma \neq 0.$$

Hence,

$$\begin{aligned} |D_\sigma^s Q_1(t,\sigma)| &= |D_\sigma^s \exp\{-t|\sigma|^\alpha\}| \\ &\le \alpha^s s! \sum_{\tilde{m}=1}^{s} t^{\tilde{m}} |\sigma|^{\alpha\tilde{m}-s} \exp\{-t|\sigma|^\alpha\} \\ &\le c_s t^{\gamma s} |\sigma|^{\omega_s - s} \exp\{-t|\sigma|^\alpha\}, \quad \sigma \neq 0, \end{aligned} \tag{8}$$

where $c_s = c_s(\alpha) > 0$,

$$\gamma = \begin{cases} 0 & \text{for} \quad 0 < t \le 1, \\ 1 & \text{for} \quad t > 1, \end{cases} \qquad \text{and} \qquad \omega_s = \begin{cases} \alpha & \text{for} \quad \sigma \neq 0, \quad |\sigma| < 1, \\ \alpha s & \text{for} \quad |\sigma| \ge 1. \end{cases}$$

By using the Leibnitz formula of differentiation of the product of two functions, we get

$$D_\sigma^s Q(t,\sigma) = D_\sigma^s (Q_1(t,\sigma) Q_2(\sigma)) = \sum_{p=0}^{s} C_s^p Q_1^{(p)}(t,\sigma) D_\sigma^{s-p} Q_2(\sigma).$$

In our subsequent analysis, we also use relation (7) with $F = \varphi^{-1}$, $\varphi = R$, where

$$R(\sigma) = \mu - \sum_{k=1}^{m} \mu_k \exp\{-t_k |\sigma|^\alpha\} = Q_2^{-1}(\sigma).$$

Then $Q_2(\sigma) = F(\varphi) = R^{-1}$ and

$$\begin{aligned} |D_\sigma^s Q_2(\sigma)| = \Bigg| \sum_{\tilde{m}=1}^{s} \frac{d^{\tilde{m}}}{dR^{\tilde{m}}} R^{-1} \sum \frac{s!}{\tilde{m}_1! \ldots \tilde{m}_l!} \\ \times \left(\frac{d}{d\sigma} R(\sigma) \right)^{\tilde{m}_1} \ldots \left(\frac{1}{l!} \frac{d^l}{d\sigma^l} R(\sigma) \right)^{\tilde{m}_l} \Bigg|, \quad \sigma \neq 0, \quad s \in \mathbb{N}. \end{aligned}$$

Let $|\sigma| \ge 1$. In view of the form of the function R and estimate (8), we obtain

$$\begin{aligned} \left| \frac{1}{j!} \frac{d^j}{d\sigma^j} R(\sigma) \right| &\le \frac{1}{j!} \sum_{k=1}^{m} \mu_k \left| \frac{d^j}{d\sigma^j} e^{-t_k |\sigma|^\alpha} \right| \\ &\le \frac{c_j}{j!} \sum_{k=1}^{m} \mu_k t_k^{\gamma j} |\sigma|^{\alpha j - j} e^{-t_k |\sigma|^\alpha} \end{aligned}$$

$$\leq c_j \sum_{k=1}^{m} \mu_k \frac{t_k^{\gamma j}}{t_k^j} |\sigma|^{-j} \equiv \beta_j |\sigma|^{-j}, \quad j \in \{1, \dots l\}$$

(here, we have used the inequality

$$|\sigma|^{\alpha j} \exp\{-t_k |\sigma|^{\alpha}\} \leq \frac{j!}{t_k^j}, \quad j \in \{1, \dots, l\} \Big).$$

Thus, if $|\sigma| \geq 1$, then the following estimate is true:

$$\Delta_0 := \left| \left(\frac{d}{d\sigma} R(\sigma) \right)^{\tilde{m}_1} \dots \left(\frac{1}{l!} \frac{d^l}{d\sigma^l} R(\sigma) \right)^{\tilde{m}_l} \right|$$

$$\leq \beta_1^{\tilde{m}_1} |\sigma|^{-\tilde{m}_1} \beta_2^{\tilde{m}_2} |\sigma|^{-2\tilde{m}_2} \dots \beta_l^{\tilde{m}_l} |\sigma|^{-l\tilde{m}_l}$$

$$\leq \beta^{\tilde{m}_1 + \dots + \tilde{m}_l} |\sigma|^{-(\tilde{m}_1 + 2\tilde{m}_2 + \dots + l\tilde{m}_l)} = \beta^{\tilde{m}} |\sigma|^{-s},$$

where $\beta = \max\{\beta_1, \dots, \beta_{\tilde{m}}\}$. If $\sigma \neq 0$, $|\sigma| < 1$, then

$$\left| \frac{1}{j!} \frac{d^j}{d\sigma^j} R(\sigma) \right| \leq c_j t_m^{\gamma j} |\sigma|^{-j} \exp\{-t_1 |\sigma|^{\alpha}\} \leq \tilde{c}_j |\sigma|^{-j}, \quad j \in \{1, \dots, l\}$$

(here we have used the inequality $0 < t_1 < \dots < t_m$). Thus, $\Delta \leq \beta^{\tilde{m}} |\sigma|^{-s}$. Moreover,

$$\frac{d^{\tilde{m}}}{dR^{\tilde{m}}} R^{-1} = (-1)^{\tilde{m}} \tilde{m}! R^{-(\tilde{m}+1)}.$$

Since $\exp\{-t_k |\sigma|^{\alpha}\} \leq 1$ $\forall \sigma \in \mathbb{R}$, $k \in \{1, \dots, m\}$, we get

$$\mu - \sum_{k=1}^{m} \mu_k \exp\{-t_k |\sigma|^{\alpha}\} \geq \mu - \sum_{k=1}^{m} \mu_k.$$

By the condition,

$$\mu > \sum_{k=1}^{m} \mu_k.$$

Hence,

$$R^{-1}(\sigma) \leq \left(\mu - \sum_{k=1}^{m} \mu_k \right)^{-1} \equiv \beta_0 > 0, \quad \left| \frac{d^{\tilde{m}}}{dR^{\tilde{m}}} R^{-1} \right| \leq \beta_0^{\tilde{m}+1} \tilde{m}!.$$

By using the last inequalities, we obtain

$$|D_\sigma^s Q_2(\sigma)| \le s! \sum_{\tilde{m}=1}^{s} \beta_0^{\tilde{m}+1} \beta^{\tilde{m}} \tilde{m}! |\sigma|^{-s} \equiv \tilde{c}_s |\sigma|^{-s}, \quad \sigma \ne 0. \tag{9}$$

Therefore [see (8) and (9)],

$$|D_\sigma^s Q(t,\sigma)| \le \sum_{p=0}^{s} C_s^p t^{\gamma p} |\sigma|^{\omega_p - p} \tilde{C}_{s-p} c_p |\sigma|^{-(s-p)} \exp\{-t|\sigma|^\alpha\}$$

$$\le b_s t^{\gamma s} |\sigma|^{\omega_s - s} \exp\{-t|\sigma|^\alpha\}, \quad \sigma \ne 0, \quad s \in \mathbb{N}.$$

Lemma 2 is proved.

Remark 2. In view of estimates (6) and (8), we get $\{Q_1(t,\cdot), Q(t,\cdot)\} \subset \Psi_\alpha$ for every $t > 0$. This fact and the boundedness of the function Q_2 on $\mathbb{R}$ also imply that Q_2 is a multiplicator in the space Q_2.

Remark 3. Reasoning similarly, we conclude that, for $t > 1$, the derivatives of the function

$$Q\left(t, t^{-1/\alpha}\sigma\right) = \exp\{-|\sigma|^\alpha\} \left(\mu - \sum_{k=1}^{m} \mu_k \exp\left\{-t^{-1} t_k |\sigma|^\alpha\right\}\right)^{-1}$$

satisfy the inequalities

$$\left|D_\sigma^s Q\left(t, t^{-1/\alpha}\sigma\right)\right| \le L_s |\sigma|^{\omega_s - s} \exp\{-|\sigma|^\alpha\}, \quad \sigma \ne 0, \quad s \in \mathbb{N}, \tag{10}$$

where the constants $L_s > 0$ are independent of t. If $0 < t \le 1$, then

$$\left|D_\sigma^s Q\left(t, t^{-1/\alpha}\sigma\right)\right| \le L_s' t^{-s} |\sigma|^{\omega_s - s} \exp\{-|\sigma|^\alpha\}, \quad \sigma \ne 0, \quad s \in \mathbb{N}. \tag{11}$$

We now consider a function

$$G(t,x) = (2\pi)^{-1} \int_{\mathbb{R}} Q(t,\sigma) e^{-ix\sigma} \, d\sigma = F^{-1}[Q(t,\sigma)].$$

It follows from Remark 2 that

$$G(t,\cdot) \in \Phi_\alpha = F^{-1}[\Psi_\alpha]$$

for every $t > 0$, i.e.,

$$|G(t,x)| \le c_k(t)(1+|x|)^{-(1+[\alpha]+k)}, \quad (t,x) \in \Omega, \quad k \in \mathbb{Z}_+.$$

In estimates for the function $G(t,x)$ and its derivatives (with respect to the variable x), we now separate the dependence on the parameter t.

Lemma 3. *The function $G(t,x)$, $(t,x) \in \Omega$, and its derivatives (with respect to the variable x) satisfy the estimates*

$$\left|D_x^k G(t,x)\right| \le c_k t^{\lambda(k)} \left(t^{1/\alpha} + |x|\right)^{-(1+[\alpha]+k)}, \quad k \in \mathbb{Z}_+, \tag{12}$$

where the constants $c_k > 0$ are independent of t,

$$\lambda(k) = \begin{cases} -((\alpha + (\alpha-1)([\alpha]+k))/\alpha) & \text{for} \quad 0 < t \le 1, \\ [\alpha]/\alpha & \text{for} \quad t > 1. \end{cases}$$

Proof. Let $k = 0$. We now perform the change of variable of integration $\sigma = t^{-1/\alpha} y$ and obtain the following representation for the function G:

$$G(t,x) = (2\pi)^{-1} t^{-1/\alpha} \int_{\mathbb{R}} Q\left(t, t^{-1/\alpha} y\right) \exp\left\{-i t^{-1/\alpha} x y\right\} dy = t^{-1/\alpha} G_0(t,z),$$

where

$$G_0(t,z) = (2\pi)^{-1} \int_{\mathbb{R}} Q\left(t, t^{-1/\alpha} y\right) \exp\{-izy\} dy, \quad z = t^{-1/\alpha} x.$$

Further, we assume that $t > 1$. If $z \neq 0$, then, as a result of integration by parts $s = 1 + [\alpha]$ times, we represent G_0 in the form

$$G_0(t,z) = (2\pi)^{-1} \lim_{\varepsilon \to +0} \int_{|y| \ge \varepsilon} Q\left(t, t^{-1/\alpha} y\right) e^{-izy} dy$$

$$= (2\pi) \frac{c^s}{z^s} \lim_{\varepsilon \to +0} \left[\int_{|y| \ge \varepsilon} D_y^s Q\left(t, t^{-1/\alpha} y\right) e^{-izy}\, dy + r(\varepsilon, z) \right],$$

where the symbol $r(\varepsilon, z)$ denotes the expression located outside the integral, which consists of terms of the form $D_y^l Q(t, t^{-1/\alpha} y) e^{-izy}$, $0 \le l \le s-1$, with values at the points $y = \varepsilon$ and $y = -\varepsilon$. In view of estimates (10), we obtain the following inequalities for $y \neq 0$, $|y| < 1$:

$$\left|D_y^l Q\left(t, t^{-1/\alpha} y\right)\right| \le L_s |y|^{\alpha - l}, \quad l \in \{1, \dots, s-1\}, \quad s = 1 + [\alpha],$$

$$\left|Q\left(t, t^{-1/\alpha} y\right)\right| \le \left(\mu - \sum_{k=1}^m \mu_k\right)^{-1}$$

and, in addition, $\alpha - l \ge \alpha - [\alpha] = \{\alpha\}$. This implies that $\lim_{\varepsilon \to +0} r(\varepsilon, z) = 0$ at every point $z \in \mathbb{R} \setminus \{0\}$. At infinity, the indicated terms located outside the integral become equal to zero because

$$\lim_{y \to \pm\infty} |D_y^l Q(t, t^{-1/\alpha} y)| = 0, \quad l \in \{0, 1, \dots, s-1\}.$$

By using the estimates for the derivatives of the function $Q(t, t^{-1/\alpha} y)$ [see (10)], we find

$$
|G_0(t,z)| \le \frac{\tilde{\tilde{c}}_s}{|z|^s} \int_0^{+\infty} y^{\omega_s - s} \exp\{-y^\alpha\}\, dy
$$

$$
= \frac{\tilde{\tilde{c}}_s}{|z|^s} \left[\int_0^1 y^{\alpha - s} \exp\{-y^\alpha\}\, dy + \int_1^{+\infty} y^{\alpha s - s} \exp\{-y^\alpha\}\, dy \right].
$$

The integrand $y^{\alpha-s} \exp\{-y^\alpha\}$ has an integrable singularity at the point $y = 0$ because $\alpha - s = \alpha - (1 + [\alpha]) = \{\alpha\} - 1$. Hence,

$$
|G_0(t,z)| \le c|z|^{-(1+[\alpha])}, \quad z \ne 0, \quad t > 1, \tag{13}
$$

and the constant $c > 0$ is independent of t. Since

$$
\left| Q_2(t^{-1/\alpha} y) \right| \le \left(\mu - \sum_{k=1}^{m} \mu_k \right)^{-1} \quad \forall t > 0, \quad y \in \mathbb{R},
$$

we get

$$
|G_0(t,z)| \le (2\pi)^{-1} \int_{\mathbb{R}} \left| Q(t, t^{-1/\alpha} y) \right| dy
$$

$$
\le (2\pi)^{-1} \left(\mu - \sum_{k=1}^{m} \mu_k \right)^{-1} \int_{\mathbb{R}} \exp\{-|y|^\alpha\}\, dy = c_0
$$

for $t > 0$ and $z \in \mathbb{R}$. By using this result and (13), we conclude that, for $t \ge 1$ and $z \in \mathbb{R}$,

$$
|G_0(t,z)| \le c_1 (1 + |z|)^{-(1+[\alpha])}, \quad z = t^{-1/\alpha} x, \quad x \in \mathbb{R}.
$$

Indeed, if $|z| \le 1$ and $|x| \le t^{1/\alpha}$, then

$$
(1 + |z|)^{1+[\alpha]} |G_0(t,z)| \le 2^{1+[\alpha]} c_0 \equiv c_1.
$$

At the same time, if $|z| > 1$ ($|x| > t^{1/\alpha}$), then, in view of estimate (13), we get

$$
(1 + |z|)^{1+[\alpha]} |G_0(t,z)| = \sum_{l=0}^{1+[\alpha]} C_{1+[\alpha]}^{l} |z|^l |G_0(t,z)|
$$

$$
\le \sum_{l=0}^{1+[\alpha]} C_{1+[\alpha]}^{l} |z|^l \frac{c}{|z|^{1+[\alpha]}} \le 2^{1+[\alpha]} c \equiv c_2.
$$

Thus,

$$|G_0(t,z)| \le \tilde{c}(1+|z|)^{-(1+[\alpha])} = \tilde{c}(1+t^{-1/\alpha}|x|)^{-(1+[\alpha])}$$

$$= \tilde{c}t^{(1+[\alpha])/\alpha}\left(t^{1/\alpha}+|x|\right)^{-(1+[\alpha])}, \quad t>1, \quad x\in\mathbb{R}.$$

This yields

$$|G(t,x)| = t^{-1/\alpha}|G_0(t,z)| \le \tilde{c}t^{[\alpha]/\alpha}\left(t^{1/\alpha}+|x|\right)^{-(1+[\alpha])}, \quad t>1, \quad x\in\mathbb{R}.$$

Let $k\in\mathbb{N}$ and $t>1$. We have

$$D_x^k G(t,x) = (2\pi)^{-1}t^{-(1+k)/\alpha}\int_{\mathbb{R}} Q\left(t,t^{-1/\alpha}y\right)y^k\exp\{-izy\}\,dy, \quad z=t^{-1/\alpha}x.$$

Reasoning as in the case $k=0$ and integrating by parts $s=1+[\alpha]+k$ times, we find

$$D_x^k G(t,x) = \frac{c_s}{z^s}t^{-(1+k)/\alpha}\int_{\mathbb{R}} D_y^s\left(Q(t,t^{-1/\alpha}y)y^k\right)e^{-izy}dy, \quad z\ne 0.$$

By using the formula of differentiation of the product of two functions, we conclude that the evaluation of derivatives of the function G is reduced to the evaluation of the sum of integrals of the form

$$\frac{|\tilde{c}_s|}{|z|^s}\left[\int_0^\infty y^k\left|D_y^s Q\left(t,t^{-1/\alpha}y\right)\right|dy + ks\int_0^\infty y^{k-1}\left|D_y^{s-1}Q\left(t,t^{-1/\alpha}y\right)\right|dy\right.$$

$$\left.+k(k-1)\frac{s(s-1)}{2!}\int_0^\infty y^{k-2}\left|D_y^{s-2}Q\left(t,t^{-1/\alpha}y\right)\right|dy+\ldots\right]. \tag{14}$$

Each integral in (14) has an integrable singularity at the point $y=0$. Indeed, consider one of the integrals of the form

$$\int_0^\infty y^{k-p}\left|D_y^{s-p}Q\left(t,t^{-1/\alpha}y\right)\right|dy, \quad 0\le p\le k, \quad s=1+[\alpha]+k, \quad t>1.$$

In view of estimates (10), the integrand admits the following estimate in a neighborhood of the point $y=0$:

$$y^{k-p}\left|D_y^{s-p}Q\left(t,t^{-1/\alpha}y\right)\right| \le L_{s-p}y^{k-p}y^{\alpha-(s-p)}$$

$$= L_s y^{k-p+\alpha-(1+[\alpha]+k-p)} = L_s y^{\alpha-[\alpha]-1} = L_s y^{\{\alpha\}-1}.$$

This yields the convergence of the corresponding integral.

As in the previous case ($k = 0$), we estimate each integral in sum (14) and arrive at the following estimate:

$$\left|D_x^k G(t,x)\right| \le c_k t^{([\alpha]+k)/\alpha} \left(t^{1/\alpha} + |x|\right)^{-(1+[\alpha]+k)}, \quad t > 1, \quad x \in \mathbb{R}.$$

Similarly, we consider the case $0 < t \le 1$ with regard for relations (11). As a result, we obtain estimates (12).
Lemma 3 is proved.

Remark 4. It follows from the properties of the function $Q(t,\sigma)$ that the function $G(t,x)$ is continuously differentiable as a function of the argument $t \in (0, +\infty)$. Moreover, since

$$Q(t,\sigma) = F[G(t,x)] = \int_{\mathbb{R}} G(t,x) e^{ix\sigma}\, dx,$$

we obtain

$$Q(t,0) = \left(\mu - \sum_{k=1}^{m} \mu_k\right)^{-1} = \int_{\mathbb{R}} G(t,x)\, dx.$$

Remark 5. Since $\mu > \sum_{k=1}^{m} \mu_k$, we obtain

$$\frac{1}{\mu} \sum_{k=1}^{m} \mu_k \exp\{-t_k|\sigma|^\alpha\} \le \frac{1}{\mu} \sum_{k=1}^{m} \mu_k < 1.$$

By using the polynomial formula, we get

$$Q_2(\sigma) = \frac{1}{\mu}\left(1 - \frac{1}{\mu}\sum_{k=1}^{m} \mu_k \exp\{-t_k|\sigma|^\alpha\}\right)^{-1} = \frac{1}{\mu}\sum_{r=0}^{\infty} \mu^{-r} \left(\sum_{k=1}^{m} \mu_k e^{-t_k|\sigma|^\alpha}\right)^r$$

$$= \sum_{r=0}^{\infty} \mu^{-(r+1)} \sum_{r_1+\ldots+r_m=r} \frac{r!\left(\mu_1 e^{-t_1|\sigma|^\alpha}\right)^{r_1} \ldots \left(\mu_m e^{-t_m|\sigma|^\alpha}\right)^{r_m}}{r_1!\ldots r_m!}$$

$$= \sum_{r=0}^{\infty} \mu^{-(r+1)} \sum_{r_1+\ldots+r_m=r} \frac{r!\mu_1^{r_1}\ldots\mu_m^{r_m}}{r_1!\ldots r_m!} Q_1(\lambda,\sigma),$$

where $\lambda := t_1 r_1 + \ldots + t_m r_m$ and $Q_1(\lambda,\sigma) = e^{-\lambda|\sigma|^\alpha}$. This yields the following representation for the function $G(t,x)$:

$$G(t,x) = (2\pi)^{-1} \int_{\mathbb{R}} \sum_{r=0}^{\infty} \mu^{-(r+1)} \sum_{r_1+\ldots+r_m=r} \frac{r!}{r_1!\ldots r_m!} \mu_1^{r_1} \ldots \mu_m^{r_m} e^{-(\lambda+t)|\sigma|^\alpha} e^{-ix\sigma}\, d\sigma$$

$$= \sum_{r=0}^{\infty} \mu^{-(r+1)} \sum_{r_1+\ldots+r_m=r} \frac{r!\mu_1^{r_1}\ldots\mu_m^{r_m}}{r_1!\ldots r_m!} \tilde{G}(t_1 r_1 + \ldots + t_m r_m + t, x),$$

where

$$\tilde{G}(\lambda + t, x) = (2\pi)^{-1} \int_{\mathbb{R}} e^{-(\lambda+t)|\sigma|^{\alpha}} e^{-ix\sigma} = F^{-1}[Q_1(\lambda + t, \sigma)].$$

Lemma 4. *The function $G(t,\cdot)$, $t \in (0, +\infty)$, as an abstract function of the parameter t with values in the space Φ_α, is differentiable with respect to t.*

Proof. Since the Fourier transform is continuous, in order to prove the lemma, it suffices to show that the function $F[G(t,\cdot)] = Q(t,\cdot)$, regarded as an abstract function of the parameter t with values in the space Ψ_α, is differentiable with respect to t. In other words, it is necessary to prove that the limit relation

$$\Gamma_{\Delta t}(\sigma) := \frac{1}{\Delta t} [Q(t + \Delta t, \sigma) - Q(t, \sigma)] \to \frac{\partial}{\partial t} Q(t, \sigma), \quad \Delta t \to 0,$$

is true in a sense of convergence in the topology of the space Ψ_α. Note that

$$\Gamma_{\Delta t}(\sigma) = -|\sigma|^{\alpha} Q(t + \theta\Delta t, \sigma), \quad 0 < \theta < 1,$$

$$\Gamma_{\Delta t}(\sigma) - \frac{\partial}{\partial t} Q(t, \sigma) = |\sigma|^{2\alpha} Q(t + \theta_1 \Delta t)\theta\Delta t, \quad 0 < \theta_1 < 1.$$

In view of the properties of the function $Q(t, \sigma)$, we can show that

$$\left\| \Gamma_{\Delta t} - \frac{\partial}{\partial t} Q \right\|_p \to 0, \quad \Delta t \to 0 \quad \forall p \in \mathbb{Z}_+.$$

Corollary 1. *The equality*

$$\frac{\partial}{\partial t} (f * G(t,\cdot)) = f * \frac{\partial G(t,\cdot)}{\partial t} \quad \forall f \in \Phi'_\alpha, \quad t > 0,$$

is true.

Proof. By the definition of convolution of a generalized function with a test function, we find

$$f * G(t, x) = \left\langle f_\xi, \check{G}(t, \xi) \right\rangle, \quad \check{G}(t, \xi) = G(t, -\xi).$$

Then

$$\frac{\partial}{\partial t}(f * G(t,\cdot)) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} [(f * G(t + \Delta t, \cdot)) - (f * G(t, \cdot))]$$

$$= \lim_{\Delta t \to 0} \left\langle f_\xi, \frac{1}{\Delta t} \left[T_{-x}\check{G}(t + \Delta t, \xi) - T_{-x}\check{G}(t, \xi) \right] \right\rangle.$$

By Lemma 4, the limit relation

$$\frac{1}{\Delta t}\left[T_{-x}\check{G}(t+\Delta t,\cdot) - T_{-x}\check{G}(t,\cdot)\right] \longrightarrow \frac{\partial}{\partial t} T_{-x}\check{G}(t,\cdot), \quad \Delta t \to 0,$$

is true in a sense of convergence in the topology of the space Φ_α. Hence, in view of the continuity of the functional f, we get

$$\frac{\partial}{\partial t}(f * G(t,x)) = \left\langle f_\xi, \lim_{\Delta t\to 0} \frac{1}{\Delta t}\left[T_{-x}\check{G}(t+\Delta t,\cdot) - T_{-x}\check{G}(t,\cdot)\right]\right\rangle$$

$$= \left\langle f_\xi, \frac{\partial}{\partial t} T_{-x}\check{G}(t,\xi)\right\rangle = \left\langle f_\xi, T_{-x}\frac{\partial}{\partial t}\check{G}(t,\xi)\right\rangle = f * \frac{\partial G(t,x)}{\partial t},$$

Q.E.D.

Lemma 5. *In the space Φ'_α, the following relations are true:*

(i) $G(t,\cdot) \to F^{-1}[Q_2]$, $t \to +0$;

(ii)

$$\mu G(t,\cdot) - \sum_{k=1}^{m} \mu_k G(t_k,\cdot) \to \delta, \quad t \to +0 \tag{15}$$

(δ is the Dirac delta function).

Proof. (i). Since the Fourier operator $F\colon \Phi'_\alpha \to \Psi'_\alpha$ is continuous, to prove the required assertion, it suffices to show that

$$F[G(t,\cdot)] = Q_1(t,\cdot)Q_2(\cdot) \to Q_2(\cdot), \quad t \to +0,$$

in the space Ψ'_α. To this end, we take an arbitrary function $\psi \in \Psi_\alpha$ and use the fact that Q_2 is a multiplicator in the space Ψ_α. Thus, by virtue of the Lebesgue theorem on limit transition under the sign of Lebesgue integral, we obtain

$$\langle Q_1(t,\cdot)Q_2(\cdot), \psi\rangle = \langle Q_1(t,\cdot), Q_2(\cdot)\psi(\cdot)\rangle$$

$$= \int_{\mathbb{R}} Q_1(t,\sigma)Q_2(\sigma)\psi(\sigma)\,d\sigma \underset{t\to+0}{\longrightarrow} \int_{\mathbb{R}} Q_2(\sigma)\psi(\sigma)\,d\sigma$$

$$= \langle 1, Q_2(\cdot)\psi(\cdot)\rangle = \langle Q_2, \psi\rangle.$$

This yields assertion (i) of Lemma 5.

(ii). By using assertion (ii) of Lemma 5, we get

$$\mu G(t,\cdot) - \sum_{k=1}^{m} \mu_k G(t_k,\cdot) \underset{t\to+0}{\longrightarrow} \mu F^{-1}[Q_2] - \sum_{k=1}^{m} \mu_k F^{-1}[Q_1(t,\cdot)Q_2(\cdot)]$$

$$= F^{-1}\left[\mu Q_2 - \sum_{k=1}^{m} \mu_k Q_1(t_k,\cdot)Q_2(\cdot)\right]$$

$$= F^{-1}\left[\left(\mu - \sum_{k=1}^{m} \mu_k Q_1(t_k,\cdot)\right) Q_2(\cdot)\right]$$

$$= F^{-1}\left[\left(\mu - \sum_{k=1}^{m} \mu_k Q_1(t_k,\cdot)\right)\left(\mu - \sum_{k=1}^{m} \mu_k Q_1(t_k,\cdot)\right)^{-1}\right] = F^{-1}[1] = \delta.$$

Thus, relation (15) is true in the space Φ'_α.

Lemma 5 is proved.

Remark 6. If $\mu = 1$ and $\mu_1 = \ldots = \mu_m = 0$, then problem (2), (3) turns into the Cauchy problem for Eq. (2). In this case, $Q_2(\sigma) = 1\ \forall\sigma \in \mathbb{R}$,

$$G(t,x) = F^{-1}[e^{-t|\sigma|^\alpha}],$$

and $G(t,\cdot) \to F^{-1}[1] = \delta$ as $t \to +0$ in the space Φ'_α.

Corollary 2. *Let*

$$\omega(t,x) = f * G(t,x), \quad f \in \Phi'_{\alpha,*}, \quad (t,x) \in \Omega$$

(*here,* $\Phi'_{\alpha,*}$ *is a class of convolvers in the space* Φ_α). *Then the following limit relation is true in the space* Φ'_α :

$$\mu\omega(t,\cdot) - \sum_{k=1}^{m} \mu_k \omega(t_k,\cdot) \to f, \quad t \to +0. \tag{16}$$

Proof. We now prove that the limit relation

$$F\left[\mu\omega(t,\cdot) - \sum_{k=1}^{m} \mu_k \omega(t_k,\cdot)\right] \to F[f], \quad t \to +0, \tag{17}$$

is true in the space Ψ'_α. Since $f \in \Phi'_{\alpha,*}$ and $G(t,\cdot) \in \Phi_\alpha$ for any $t > 0$, we obtain

$$F[\omega(t,\cdot)] = F[f * G(t,\cdot)] = F[f]\cdot F[G(t,\cdot)] = F[f]\cdot Q(t,\cdot).$$

Hence, it is necessary to show that

$$F[f]\left(\mu Q(t,\cdot)-\sum_{k=1}^{m}\mu_k Q(t_k,\cdot)\right)\to F[f]$$

as $t\to+0$ in the space Ψ'_α. Since

$$Q(t,\cdot)=Q_1(t,\cdot)Q_2(\cdot)\to Q_2(\cdot)\qquad \text{as}\quad t\to+0$$

in the space Ψ'_α (see the proof of assertion (i) in Lemma 5), we conclude that

$$\mu Q(t,\cdot)-\sum_{k=1}^{m}\mu_k Q(t_k,\cdot)\underset{t\to+0}{\longrightarrow}\mu Q_2(\cdot)-\sum_{k=1}^{m}\mu_k Q_1(t_k,\cdot)Q_2(\cdot)$$

$$=\left(\mu-\sum_{k=1}^{m}\mu_k Q_1(t_k,\cdot)\right)Q_2(\cdot)=1$$

in the space Ψ'_α.

Thus, relation (17) and, hence, also (16) are true in the corresponding spaces.

Corollary 2 is proved.

Remark 7. The function $G(t,x)$, $(t,x)\in\Omega$, is a solution of Eq. (2). Indeed,

$$\frac{\partial}{\partial t}G(t,x)=-F^{-1}\left[|\sigma|^{\alpha}Q(t,\sigma)\right],$$

$$\hat{A}G(t,x)=F^{-1}\left[|\sigma|^{\alpha}F[F^{-1}Q(t,\cdot)]\right]=F^{-1}\left[|\sigma|^{\alpha}Q(t,\sigma)\right].$$

This yields

$$\frac{\partial}{\partial t}G(t,x)+\hat{A}G(t,x)=0,\quad (t,x)\in\Omega,$$

Q.E.D.

In what follows, we say that the function $G(t,x)$, $(t,x)\in\Omega$, is a *fundamental solution* of the multipoint (in time) problem for Eq. (2).

By Corollary 2, the nonlocal multipoint (in time) problem for Eq. (2) can be formulated as follows: To find a function $u(t,x)$, $(t,x)\in\Omega$, satisfying Eq. (2) and the condition

$$\mu\lim_{t\to+0}u(t,\cdot)-\sum_{k=1}^{m}\mu_k u(t_k,\cdot)=f,\quad f\in\Phi'_{\alpha,*}\tag{18}$$

[the limit relation (18) is considered in the space Φ'_α and the restrictions imposed on the parameters $\mu,\mu_1,\ldots,\mu_m$, $t_1,\ldots,t_m$ are the same as in problem (2), (3)].

Theorem 1. *The nonlocal multipoint (in time) problem (2), (18) is correctly solvable, its solution is given by the formula*

$$u(t,x) = f * G(t,x), \quad (t,x) \in \Omega,$$

and $u(t,\cdot) \in \Phi_\alpha$ *for any* $t > 0$.

Proof. We now show that the function $u(t,x)$, $(t,x) \in \Omega$, satisfies Eq. (2). Indeed (see, Corollary 1),

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial}{\partial t}(f * G(t,x)) = f * \frac{\partial G(t,x)}{\partial t}$$

and

$$\hat{A}u(t,x) = F^{-1}\big[|\sigma|^\alpha F[f * G(t,\cdot)]\big].$$

Since f is a convolver in the space Φ_α, we get

$$F[f * G(t,\cdot)] = F[f]F[G(t,\cdot)] = F[f]Q(t,\cdot).$$

Hence,

$$\hat{A}u(t,x) = F^{-1}\big[|\sigma|^\alpha Q(t,\sigma)F[f]\big] = -F^{-1}\left[\frac{\partial}{\partial t}Q(t,\cdot)F[f]\right]$$

$$= -F^{-1}\left[F\left[\frac{\partial}{\partial t}G(t,\cdot)\right]F[f]\right] = -F^{-1}\left[F\left[f * \frac{\partial G(t,\cdot)}{\partial t}\right]\right] = -f * \frac{\partial G(t,\cdot)}{\partial t}.$$

This implies that the function $u(t,x)$, $(t,x) \in \Omega$, satisfies Eq. (2).

It follows from Corollary 2 that u satisfies condition (18) in the indicated sense. We also note that u continuously depends on the function $f \in \Phi'_{\alpha,*}$ because the operation of convolution has the property of continuity.

It remains to show that problem (2), (18) possesses a unique solution. To this end, we consider the Cauchy problem

$$\frac{\partial v(t,x)}{\partial t} = \hat{A}^* v(t,x), \quad (t,x) \in [0,t_0) \times \mathbb{R} \equiv \Omega', \quad 0 \le t < t_0 < +\infty, \tag{19}$$

$$v(t,\cdot)\Big|_{t=t_0} = \psi, \quad \psi \in \Phi'_{\alpha,*}, \tag{20}$$

where $\hat{A}^*$ is the restriction of the adjoint operator $\hat{A}$ to the space Φ_α. Condition (20) is understood in a weak sense. The Cauchy problem (19), (20) is correctly solvable. Its solution is given by the formula

$$v(t,x) = \psi * G^*(t,x), \quad G^*(t,x) = F^{-1}[\exp\{(t-t_0)|\sigma|^\alpha\}], \quad v(t,\cdot) \in \Phi_\alpha$$

for every $t \in [0,t_0)$.

Let $Q_{t_0}^t\colon \Phi'_{\alpha,*} \to \Phi_\alpha$ be an operator that associates the functional $\psi \in \Phi'_{\alpha,*}$ with the solution of problem (19), (20). The operator $Q_{t_0}^t$ is linear and continuous. It is defined for any t and t_0 such that $0 \le t < t_0 < +\infty$, and has the following properties:

$$\forall \psi \in \Phi'_{\alpha,*}\colon \frac{dQ_{t_0}^t \psi}{dt} = \hat{A}^* Q_{t_0}^t \psi, \quad \lim_{t \to t_0} Q_{t_0}^t \psi = \psi,$$

(the limit is considered in the space Φ'_α).

Consider a solution $u(t,x)$, $(t,x) \in \Omega$, of problem (2), (18), which is understood as a regular functional from the space $\Phi'_{\alpha,*} \supset \Phi_\alpha$. We now prove that problem (2), (18) may have only a unique solution in the space $\Phi'_{\alpha,*}$. To this end, it suffices to show that the role of unique solution of Eq. (2) with the trivial initial condition can be played solely by the functional $u(t,x) = 0$ (for every $t \in (0,\infty)$). We apply the functional u to a function $Q_{t_0}^t \psi \in \Phi_\alpha$, where ψ is an arbitrary fixed element from the space $\Phi_\alpha \subset \Phi'_{\alpha,*}$. Differentiating with respect to t, in view of Eqs. (2) and (19), we get

$$\begin{aligned}\frac{\partial}{\partial t} \langle u(t,\cdot), Q_{t_0}^t \psi \rangle &= \left\langle \frac{\partial u}{\partial t}, Q_{t_0}^t \psi \right\rangle + \left\langle u, \frac{\partial Q_{t_0}^t \psi}{\partial t} \right\rangle \\ &= \left\langle -\hat{A}u,\ Q_{t_0}^t \psi \right\rangle + \left\langle u, \hat{A}^*\ Q_{t_0}^t \psi \right\rangle \\ &= -\left\langle \hat{A}u,\ Q_{t_0}^t \psi \right\rangle + \langle Au,\ Q_{t_0}^t \psi \rangle = 0, \quad t \in [0, t_0).\end{aligned}$$

Thus, $\langle u(t,\cdot), Q_{t_0}^t \psi \rangle$ is a constant. By using the properties of abstract functions, we obtain the relation

$$\lim_{t \to t_0} \langle u(t,\cdot), Q_{t_0}^t \psi \rangle = \langle u(t_0,\cdot), \psi \rangle = \text{const} \equiv c, \quad c = c(t_0),$$

at any point $t_0 \in (0,+\infty)$. Hence, if $f = 0$ in (18), then

$$\mu \lim_{t \to +0} \langle u(t,\cdot), \psi \rangle - \sum_{k=1}^m \mu_k \langle u(t_k,\cdot), \psi \rangle = \mu c_0 - \sum_{k=1}^m \mu_k c_k = 0.$$

This implies that $c_0 = c_1 = \ldots = c_m = 0$. Indeed, assume that this is not true. Thus, let $c_0 \neq 0$. This yields the relation $\mu - \sum_{k=1}^m \mu_k \alpha_k = 0$, where $\alpha_k = c_k / c_0$, i.e.,

$$\mu = \sum_{k=1}^m \mu_k \alpha_k.$$

Since α_k are arbitrary constants and, by the condition, $\mu, \mu_1, \ldots, \mu_m$ are fixed parameters such that

$$\mu > \sum_{k=1}^m \mu_k,$$

the obtained contradiction proves that $c_0 = 0$. Similarly, we can show that $c_1 = \ldots = c_m = 0$.

Thus, $\langle u(t_0,\cdot),\psi\rangle = 0$ for any $\psi \in \Phi_\alpha$, i.e., $u(t_0,x)$ is a null functional from the space $\Phi'_{\alpha,*}$. Since $t_0 \in (0,+\infty)$ and t_0 is chosen arbitrarily, we conclude that $u(t,x) = 0$ for all $t \in (0,+\infty)$.

Theorem 1 is proved.

Theorem 2. *Suppose that $u(t,x)$, $(t,x) \in \Omega$, is a solution of problem (2), (18) with an initial function $f \in \Phi'_{\alpha,*}$ with bounded support (i.e.,* supp f *is a bounded set in $\mathbb{R}$). Then $u(t,x) \to 0$ as $t \to +\infty$ uniformly in $\mathbb{R}$.*

Proof. Let supp $f \subset [a_1,b_1] \subset [a_2,b_2] \subset \mathbb{R}$. Consider a function $\varphi \in \Phi_\alpha$ such that $\varphi(x) = 1$, $x \in [a_1,b_1]$, and supp $\varphi \subset [a_2,b_2]$. This function exists because the space Φ_α contains finite functions. We represent the function $u(t,x)$ in the form

$$u(t,x) = \left\langle f_\xi, \varphi(\xi)G(t,x-\xi)\right\rangle + \left\langle f_\xi, \gamma(\xi)G(t,x-\xi)\right\rangle,$$

where $\gamma = 1-\varphi$. Since supp $(\gamma(\xi)G(t,x-\xi)) \cap$ supp $f = \varnothing$, we have

$$u(t,x) = t^{-1/\alpha}\left\langle f_\xi, t^{1/\alpha}\varphi(\xi)G(t,x-\xi)\right\rangle.$$

The generalized function

$$f \in \Phi'_{\alpha,*} \subset \Phi'_\alpha = \bigcup_{p=0}^{\infty} \Phi'_{p,\alpha}$$

has a finite order. Hence,

$$|u(t,x)| \le t^{-1/\alpha}\|f\|_p \|\Gamma_{t,x}\|_p,$$

where

$$\Gamma_{t,x}(\xi) = t^{1/\alpha}\varphi(\xi)G(t,x-\xi).$$

Note that $\Gamma_{t,x}(\xi) = 0$ for $\xi \in \mathbb{R}\setminus[a_2,b_2]$. Therefore, in order to prove the formulated assertion, it suffices to show that $\Gamma_{t,x}(\xi)$ is bounded in the norm of the space $\Phi_{p,\alpha}$, i.e., $\|\Gamma_{t,x}\|_p \le c_p$, where the constant $c_p > 0$ is independent of t and x ($t > 1$ and $x \in \mathbb{R}$). To this end, we use the estimate

$$\begin{aligned}\left|D^l_\xi G(t,x-\xi)\right| &\le c_l t^{[\alpha]/\alpha}\left(t^{1/\alpha} + |x-\xi|\right)^{-(1+[\alpha]+l)} \\ &\le c_l t^{[\alpha]/\alpha} t^{-(1+[\alpha]+l)/\alpha} \le c_l t^{-1/\alpha}, \quad l \in \mathbb{Z}_+, \end{aligned} \tag{21}$$

which is true for $t > 1$, $x \in \mathbb{R}$, and $\xi \in [a_2,b_2]$ and follows from (12). Since $\Gamma_{t,x}(\xi) = 0$ for $\xi \in \mathbb{R}\setminus[a_2,b_2]$, by virtue of (21), we get

$$\|\Gamma_{t,x}\|_p = t^{1/\alpha} \sup_{\xi\in[a_2,b_2]} \left\{\sum_{k=0}^{p}(1+|\xi|)^{1+[\alpha]+k}\left|(\varphi(\xi)G(t,x-\xi))^{(k)}\right|\right\}$$

$$\leq t^{1/\alpha} \sup_{\xi \in [a_2, b_2]} \left\{ \sum_{k=0}^{p} (1+c)^{1+[\alpha]+k} \sum_{l=0}^{k} C_k^l \left| D_\xi^l \varphi(\xi) \right| \left| D_\xi^l G(t, x-\xi) \right| \right\} \leq c_p,$$

where $c = \max\{|a_2|, |b_2|\}$ (here, we have used the fact that $|\varphi^{(l)}(\xi)| \leq c'_l$, $l \in \{0, 1, \ldots, k\}$, $\xi \in [a_2, b_2]$). Thus,

$$|u(t,x)| \leq \tilde{c}_p t^{-1/\alpha}, \quad t > 1, \quad x \in \mathbb{R},$$

where $\tilde{c}_p = c_p \|f\|_p$, which implies that $u(t,x) \to 0$ as $t \to +\infty$ uniformly on $\mathbb{R}$.

Theorem 2 is proved.

In particular, if $f = \delta \in \Phi'_{\alpha,*}$ and $\operatorname{supp}\delta = \{0\}$ in condition (18), then

$$u(t,x) = \delta * G(t,x) = G(t,x) \to 0$$

as $t \to +\infty$ uniformly on $\mathbb{R}$. In this case, the same result directly follows either from estimate (12) (for $c = 0$) or from the estimate

$$|G(t,x)| = (2\pi)^{-1} \left| \int_{\mathbb{R}} Q(t,\sigma) e^{-i\sigma x} d\sigma \right| \leq (2\pi)^{-1} \int_{\mathbb{R}} |Q(t,\sigma)| d\sigma$$

$$\leq (2\pi)^{-1} \int_{\mathbb{R}} e^{-t|\sigma|^\alpha} |Q_2(\sigma)|\, d\sigma$$

$$\leq (2\pi)^{-1} \left(\mu - \sum_{k=1}^{m} \mu_k \right)^{-1} \int_{\mathbb{R}} e^{-t|\sigma|^\alpha}\, d\sigma$$

$$= c_0 t^{-1/\alpha} \int_{\mathbb{R}} e^{-|y|^\alpha}\, dy = c'_0 t^{-1/\alpha} \quad \forall x \in \mathbb{R}.$$

4. Conclusions

It is shown that the restriction of the operator $A = |D_x|^\alpha$, $\alpha \in (1, +\infty) \setminus \{2, 3, \ldots\}$ to the space Φ_α coincides with the pseudodifferential operator constructed according to the function symbol $\chi(\sigma) = |\sigma|^\alpha$, $\sigma \in \mathbb{R}$, which is not differentiable at the point $\sigma = 0$. This enables us to apply the method of Fourier transformation to the investigation of the nonlocal multipoint problem for the evolutionary equation with this operator. We prove the correct solvability of the indicated problem with initial function, which is an element of the space of generalized functions of the distribution type. The proposed statement of the problem enables one to extend the class of initial functions because each function with power singularity at the point 0 can be regularized in the space of Schwarz-type distributions (i.e., it can be regarded as a regular functional). We also obtain the representation of the solution in the form of the convolution of the fundamental solution with the initial function, investigate the properties of this fundamental solution, and establish the fact that, under certain restrictions imposed on the initial function, the solution of the problem uniformly converges to zero as $t \to +\infty$ on $\mathbb{R}$.

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