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MULTIPOINT BOUNDARY-VALUE PROBLEM OF OPTIMAL CONTROL FOR PARABOLIC EQUATIONS WITH DEGENERATION

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We study the problem of optimal control over the process described by a multipoint problem with oblique derivative for a second-order parabolic equation and consider the cases of internal, starting, and boundary control. The performance criterion is specified as the sum of volume and surface integrals. By using the principle of maximum and *a priori* estimates, we establish the existence and uniqueness of solutions of a multipoint boundary-value problem with degeneration. The coefficients of the parabolic equation and boundary conditions have power singularities of any order for any variables on a certain set of points. We establish estimates for the solution of the multipoint boundary-value problem and its derivatives in Hölder spaces with power weight. The necessary and sufficient conditions for the existence of the optimal solution of the problem are presented.

Keywords: interpolation inequalities, maximum principle, *a priori* estimates, degeneration, boundary conditions.

The theory of optimal control over the processes described by the boundary-value problems for partial differential equations is characterized by a great number of accumulated results. Moreover, it is extensively developed at present. The necessity of investigations of this kind is connected with their active applications in solving the problems of natural sciences including, in particular, the problems of hydrodynamics and gas dynamics, physics of heat, filtration, diffusion, plasma, and the theory of biological populations.

For the first time, fundamentals of the theory of optimal control of deterministic systems governed by partial differential equations, were systematically described in the monograph [3]. Important results obtained in this theory for the case of evolutionary equations given in a bounded time interval were obtained, in particular, in [1, 11–13, 15]. In [14, 16], the state of controlled systems was described by the Dirichlet problem for linear parabolic equations. The existence and uniqueness of the optimal control in the case of final observation was proved and necessary conditions of optimality were established in the form of generalized Lagrange multipliers rule in [16].

The works [5, 8] were devoted to the problem of choice of the optimal control over the processes described by parabolic boundary-value problems with bounded internal control. The performance functional has the form of a volume integral.

In the present paper, we consider the multipoint boundary-value problem with oblique derivative for a parabolic equation with power singularities of any order in coefficients of the equation and in the boundary condition with respect to any variables on a certain set of points. By using *a priori* estimates and the maximum principle, we prove the existence of unique solution of the posed problem and establish estimates for its derivatives in Hölder spaces with power weights. The accumulated results are used to establish necessary and sufficient con-

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ditions for the existence of the optimal solution to a system described by the boundary-value problem with internal, starting, and boundary controls and integral performance criteria.

1. Statement of the Problem and Main Restrictions

Let $t_0, t_1, \ldots, t_{N+1}, \eta$ be arbitrary numbers, $0 \le t_0 < t_1 < \ldots < t_{N+1}$, $\eta \ne t_k$, $k \in \{0, 1, \ldots, N+1\}$, let D be a bounded domain in $\mathbb{R}^n$ with boundary ∂D, $\dim D = n$, and let Ω be a bounded domain, $\bar{\Omega} \subset D$ with $\dim \Omega \le n-1$. Denote

$$Q_{(0)} = \{(t,x) | t \in [t_0, t_{N+1}], x \in \Omega\} \cup \{(t,x) | t = \eta, x \in D\}, \quad \eta \in (t_0, t_{N+1}), \quad \Gamma = [0,T) \times \partial D.$$

In the domain $Q = [t_0, t_{N+1}) \times D$, we consider the problem of finding the functions (u, q), $q = (q_1, q_2, q_3)$ for which the functional

$$I(q) = \int_{t_0}^{t_{N+1}} dt \int_D F_1(t, x; u(t, x; q), q_1(t, x))\, dx$$

$$+ \int_{t_0}^{t_{N+1}} dt \int_{\partial D} F_2(t, x; u(t, x; q), q_2(t, x))\, d_x S$$

$$+ \int_D F_3(x; u(t_1, x; q), u(t_2, x; q), \ldots, u(t_N, x; q) q_3(x))\, dx \tag{1}$$

attains its minimum in the class of the functions

$$q \in V = \{q | q_1 \in C^{\alpha}(Q), q_2 \in C^{1+\alpha}(D), q_3 \in C^{2+\alpha}(Q), \nu_{11}(t,x) \le q_1$$

$$\le \nu_{12}(t,x), \nu_{21}(t,x) \le q_2 \le \nu_{22}(t,x), \nu_{31}(x) \le q_3 \le \nu_{32}(x)\},$$

where $\alpha \in (0,1)$, C^{α} is the space of functions with continuous derivative of order α; q_1, q_2, and q_3 are controls (internal, boundary, and limit), and $\nu_{mn}(t,x)$ is the restriction imposed on control. In this case, for $(t,x) \in Q^{(0)} = Q \setminus Q_{(0)}$, the function $u(t, x; q_1(t,x), q_2(t,x), q_3(x))$ satisfies the equation

$$(Lu)(t,x) \equiv \left[\partial_t - \sum_{i,j=1}^{n} A_{ij}(t,x) \partial_{x_i} \partial_{x_j} + \sum_{i=1}^{n} A_i(t,x) \partial_{x_i} + A_0(t,x) \right] u$$

$$= f(t, x; q_1(t,x)), \tag{2}$$

the nonlocal condition with respect to the time variable

$$(Bu)(x) \equiv u(t_0, x; q) + \sum_{k=1}^{N} \zeta_k(x) u(t_k, x; q) = \varphi(x; q_3(x)), \tag{3}$$

and the boundary condition

$$\lim_{x \to z \in \partial D} (B_1 u - \psi)(t, x) = \lim_{x \to z \in \partial D} \left[\sum_{k=1}^{n} b_k(t, x) \partial_{x_k} u(t, x; q) - b_0(t, x) u(t, x; q) - \psi(t, x; q_3(x)) \right] = 0 \tag{4}$$

on the lateral surface.

The power singularities of the coefficients of differential expressions L and B_1 at the point $P(t,x) \in Q^{(0)}$ are characterized by the functions $s_1(\beta_i^{(1)}, t)$ and $s_2(\beta_i^{(2)}, x)$:

$$s_1(\beta_i^{(1)}, t) = \begin{cases} |t - \eta|^{\beta_i^{(1)}}, & |t - \eta| \le 1, \\ 1, & |t - \eta| > 1, \end{cases} \qquad s_2(\beta_i^{(2)}, x) = \begin{cases} \rho^{\beta_i^{(2)}}(x), & \rho(x) \le 1, \\ 1, & \rho(x) > 1, \end{cases}$$

$$\rho(x) = \inf_{z \in \Omega} |x - z|, \qquad \beta_i^{(\nu)} \in (-\infty, \infty), \qquad \nu \in \{1,2\}, \qquad \beta^{(\nu)} = (\beta_1^{(\nu)}, \ldots, \beta_n^{(\nu)}), \qquad \beta = (\beta^{(1)}, \beta^{(2)}).$$

We now define the spaces used to study problem (1)–(4). Denote by ℓ, $q^{(1)}$, $q^{(2)}$, $\gamma^{(1)}$, $\gamma^{(2)}$, $\delta^{(1)}$, $\delta^{(2)}$, $\mu_j^{(1)}$, and $\mu_j^{(2)}$ real numbers, $j \in \{0,1,\ldots,n\}$, $\ell \ge 0$, $[\ell]$ is the integral part of a number ℓ, $\{\ell\} = \ell - [\ell]$, $q^{(\nu)} \ge 0$, $\gamma^{(\nu)} \ge 0$, $\delta^{(\nu)} \ge 0$, $\mu_j^{(\nu)} \ge 0$, $\nu \in \{1,2\}$; $P(t,x)$, $P_1(t^{(1)}, x^{(1)})$, $P_2(t^{(2)}, x^{(1)})$, $R_i(t^{(1)}, x^{(2)})$ are arbitrary points from $Q^{(0)}$, $i \in \{1,2,\ldots,n\}$, $x^{(1)} = (x_1^{(1)}, \ldots, x_i^{(1)}, \ldots, x_n^{(1)})$, and $x^{(2)} = (x_1^{(1)}, \ldots, x_{i-1}^{(1)}, x_i^{(2)}, x_{i+1}^{(1)} \ldots, x_n^{(1)})$.

By $H^{\ell}(\gamma; \beta; q; Q)$ we denote the set of functions u with continuous derivatives in $Q^{(0)}$ of the form $\partial_t^s \partial_x^r u$, $2s + |r| \le [\ell]$, for which the norm

$$\|u; \gamma; \beta; 0; Q\|_0 = \{\sup_{\bar{Q}} |u|\} \equiv \|u; Q\|_0,$$

$$\|u; \gamma; \beta; q; Q\|_\ell = \sum_{2s+|r| \le [\ell]} \|u; \gamma; \beta; q; Q\|_{2s+|r|} + \langle u; \gamma; \beta; q; Q \rangle_\ell,$$

is finite. Here, e.g., we have

$$\|u; \gamma; \beta; q; Q\|_{2s+|r|} \equiv \sup_{P \in \bar{Q}} \left[s_1(q^{(1)} + 2s\gamma^{(1)}, t) s_2(q^{(2)} + 2s\gamma^{(2)}, x) \left| \partial_t^s \partial_x^r u(P) \right| \right.$$

$$\times \prod_{i=1}^{n} s_1(r_i(\gamma^{(1)} - \beta_i^{(1)}), t) s_2(r_i(\gamma^{(2)} - \beta_i^{(2)}), x) \Bigg],$$

$$\langle u; \gamma; \beta; q; Q \rangle_\ell \equiv \sum_{2s+|r|=[\ell]} \Bigg\{ \sum_{\nu=1}^{n} \Bigg[\sup_{(P_2, R_\nu) \subset Q} \Bigg[s_1(q^{(1)} + 2s\gamma^{(1)}, t^{(2)})$$

$$\times s_2(q^{(2)} + 2s\gamma^{(2)}, \tilde{x}) \prod_{i=1}^{n} s_1(r_i(\gamma^{(1)} - \beta_i^{(1)}), t^{(2)})$$

$$\times s_2(r_i(\gamma^{(2)} - \beta_i^{(2)}), \tilde{x}) \left| \partial_t^s \partial_x^r u(P_2) - \partial_t^s \partial_x^r u(R_\nu) \right|$$

$$\times \left| x_\nu^{(1)} - x_\nu^{(2)} \right|^{-\{\ell\}} s_1(\{\ell\}(\gamma^{(1)} - \beta_\nu^{(1)}), t^{(2)})$$

$$\times s_2(\{\ell\}(\gamma^{(2)} - \beta_\nu^{(2)}), \tilde{x}) \Bigg] + \sup_{(P_1, P_2) \subset \bar{Q}} \Bigg[s_1(q^{(1)} + \ell\gamma^{(1)}, \tilde{t})$$

$$\times s_2(q^{(2)} + (2s + \{\ell\})\gamma^{(2)}, x^{(1)}) \prod_{i=1}^{n} s_1(-r_i\beta_i^{(1)}, \tilde{t})$$

$$\times s_2(r_i(\gamma^{(2)} - \beta_i^{(2)}), x^{(1)}) \left| t^{(1)} - t^{(2)} \right|^{-\{\ell/2\}}$$

$$\times \left| \partial_t^s \partial_x^r u(P_1) - \partial_t^s \partial_x^r u(P_2) \right| \Bigg] \Bigg\}, \qquad |r| = r_1 + \ldots + r_n,$$

$$s_1(q, \tilde{t}) = \min\{s_1(q, t^{(1)}), s_2(q, t^{(2)})\}, \quad \text{and} \quad s_2(q, \tilde{x}) = \min\{s_2(q, x^{(1)}), s_2(q, x^{(2)})\}.$$

Suppose that, for problem (1)–(4), the following conditions are satisfied:

(1°) for any vector $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ and $\forall (t, x) \in Q \setminus Q_{(0)}$, the inequality

$$\pi_1 |\xi|^2 \le \sum_{i,j=1}^{n} A_{ij}(t, x) s_1(\beta_i^{(1)}, t) s_1(\beta_j^{(1)}, t) s_2(\beta_i^{(2)}, x) s_2(\beta_j^{(2)}, x) \xi_i \xi_j \le \pi_2 |\xi|^2, \tag{5}$$

holds; here, π_1 and π_2 are fixed positive constants,

$$s_1(\beta_i^{(1)}, t) s_1(\beta_j^{(1)}, t) s_2(\beta_i^{(2)}, x) s_2(\beta_j^{(2)}, x) A_{ij} \in H^\alpha(\gamma; \beta; 0; Q),$$

$$s_1(\mu_i^{(1)}, t) s_2(\mu_i^{(2)}, x) A_i \in H^\alpha(\gamma; \beta; 0; Q),$$

$$s_1(\mu_0^{(1)},t)s_2(\mu_0^{(2)},x)A_0 \in H^{\alpha}(\gamma;\beta;0;Q), \qquad A_0 \geq 0,$$

$$s_1(\delta^{(1)},t)s_2(\delta^{(2)},x)b_0 \in H^{1+\alpha}(\gamma;\beta;0;Q),$$

$$s_1(\beta_i^{(1)},t)s_2(\beta_i^{(2)},x)b_i \in H^{1+\alpha}(\gamma;\beta;0;Q),$$

the vectors

$$\mathbf{b}^{(s)} = \{b_1^{(s)},\dots,b_n^{(s)}\},$$

where $b_i^{(s)} = s_1(\beta_i^{(1)},t)s_2(\beta_i^{(2)},x)b_i$, and $\mathbf{e} = \{e_1,\dots,e_n\}$, $e_i = b_i\left(\sum_{k=1}^{n} b_k^2\right)^{-1/2}$, form an angle smaller than $\pi/2$, with the direction of the outer normal $\mathbf{n}$ to ∂D at the point $P(t,x) \in \Gamma$, $b_0(t,x)|_{\Gamma} > 0$, $\partial D \in C^{2+\alpha}$, and

$$\lim_{x \to z \in \partial D}\left[\psi(t_0,x;q_2(t_0,x)) + \sum_{k=1}^{N}\zeta_k(x)\psi(t_k,x;q_2(t_k,x)) - B_1\varphi(x;q_3(x))\right] = 0,$$

$$\lim_{x \to z \in \partial D}\left(\sum_{i=1}^{n} b_i(t,x)\frac{\partial \zeta_k(x)}{\partial x_i}\right) = 0, \qquad \sum_{k=1}^{N}|\zeta_k(x)| \leq \lambda_0 < 1,$$

$$\zeta_k(x) \in C^{2+\alpha}(D);$$

(2°)

$$\nu_{11} \in C^{\alpha}(Q), \qquad \nu_{12} \in C^{\alpha}(Q), \qquad f(t,x;q_1(t,x)) \equiv F(t,x) \in H^{\alpha}(\gamma;\beta;0;Q),$$

$$\nu_{31} \in C^{2+\alpha}(D), \qquad \nu_{32} \in C^{2+\alpha}(D), \qquad \varphi(x;q_3(x)) \equiv \Phi(x) \in H^{2+\alpha}(\tilde{\gamma};\tilde{\beta};0;D),$$

$$\nu_{21} \in C^{1+\alpha}(Q), \qquad \nu_{22} \in C^{1+\alpha}(Q), \qquad \psi(t,x;q_2(t,x)) \equiv G(t,x) \in H^{1+\alpha}(\gamma;\beta;0;Q),$$

$$\tilde{\beta} = (0,\beta^{(2)}), \qquad \tilde{\gamma} = (0,\gamma^{(2)}),$$

$$\gamma^{(\nu)} = \max\left\{\max_i(1+\beta_i^{(\nu)}), \max_i(\mu_i^{(\nu)} - \beta_i^{(\nu)}), \frac{\mu_0^{(\nu)}}{2}, \delta^{(\nu)}\right\}, \qquad \nu \in \{1,2\}.$$

The following theorem is true:

Theorem 1. *Suppose that conditions (1°) and (2°) are satisfied for problem (2)–(4). Then there exists a unique solution of problem (2)–(4) from the space* $H^{2+\alpha}(\gamma;\beta;0;Q)$, *and the inequality*

$$\|u;\gamma;\beta;0;Q\|_{2+\alpha} \le c(\|f;\gamma;;0;Q\|_{\alpha} + \left\|\varphi;\tilde{\gamma};\tilde{\beta};0;D\right\|_{2+\alpha} + \|\psi;\gamma;\beta;0;Q\|_{1+\alpha}) \tag{6}$$

is true.

To prove Theorem 1, we first establish the correct solvability of boundary-value problems with smooth coefficients. In the obtained family of solutions, we select a convergent sequence whose limit value is the solution of problem (2)–(4).

2. Estimation of the Solutions of Boundary-Value Problems with Smooth Coefficients

Let $Q_m = Q \cup \{(t,x) \in Q \,|\, s_1(1,t) \ge m_1^{-1}, s_2(1,x) \ge m_2^{-1}\}$, $m = (m_1, m_2)$, $m_1 > 1$, $m_2 > 1$, be sequences of domains that converge to Q as $m_1 \to \infty$ and $m_2 \to \infty$.

In the domain Q, we consider the problem of finding the solutions of the equation

$$(L_1 u_m)(t,x) \equiv \left[\partial_t - \sum_{i,j=1}^{n} a_{ij}(t,x)\partial_{x_i}\partial_{x_j} + \sum_{i=1}^{n} a_i(t,x)\partial_{x_i} + a_0(t,x)\right] u_m(t,x)$$

$$= f_m(t,x;q_1) \tag{7}$$

satisfying the following conditions with respect to the variable t:

$$(Bu_m)(x) = \varphi_m(x;q_3) \tag{8}$$

and the boundary condition

$$\lim_{x\to z\in\partial D}(B_2 u_m - \psi_m)(t,x) \equiv \lim_{x\to z\in\partial D}\left[\sum_{i=1}^{n} h_i(t,x)\partial_{x_i} u_m\right.$$

$$\left. + h_0(t,x)u_m - \psi_m(t,x;q_3)\right] = 0. \tag{9}$$

Here, the coefficients a_{ij}, a_i, a_0, h_i, and h_0 and the functions f_m, φ_m, and ψ_m are determined as follows: If $(t,x) \in Q_m$, then the coefficients a_{ij}, a_i, a_0, h_i, and h_0 and the functions f_m, φ_m, and ψ_m coincide with A_{ij}, A_i, A_0, b_i, b_0, f, φ, and ψ, respectively. Moreover, in the domains $Q \setminus Q_m$, they are continuous extensions of the coefficients A_{ij}, A_i, A_0, b_i, and b_0 and the functions f, φ, and ψ, respectively, from the domains Q_m to the domains $Q \setminus Q_m$ with preservation of smoothness and the norm (see [10, p. 82].

Theorem 2. *Suppose that* $u_m(t,x)$ *is a classical solution of problem (7)–(9) in the domain* Q *and that conditions (1°) and (2°) are satisfied. Then the following estimate is true for* $u_m(t,x)$:

$$\|u_m;Q\|_0 \le c(\|\varphi_m;D\|_0 + \|f_m;Q\|_0 + \|\psi_m;Q\|_0). \tag{10}$$

Proof. The proof of estimate (10) is obtained by using the same methods as in the proof of Theorem 2.2 in [2, p. 25], i.e., we analyze all possible positions of the positive maximum and negative minimum of the function $u_m(t,x)$.

By $(G_m^{(1)}(t,x,\tau,\xi), G_m^{(2)}(t,x,\tau,\xi))$ we denote the Green function [4, p. 141] of the boundary-value problem

$$(L_1 u_m)(t,x) = f_m(t,x;q_1),$$

$$u_m(0,x) = \varphi_m(x;q_3), \tag{11}$$

$$\lim_{x\to z\in\partial D}(B_2 u_m - \psi_m)(t,x) = 0.$$

Theorem 3. *Suppose that conditions (1°) and (2°) are satisfied. Then there exists the Green function of problem (7)–(9) with the components* $\{G_m^{(1)}, G_m^{(2)}, Z_1^{(1)},\ldots,Z_N^{(1)}, Z_1^{(2)},\ldots,Z_N^{(2)}\}$ *and the following formula is true:*

$$u_m(t,x) = \int_0^t d\tau \int_D G_m^{(1)}(t,x,\tau,\xi) f_m(\tau,\xi;q_1)\,d\xi + \int_D G_m^{(1)}(t,x,0,\xi)\varphi_m(\xi;q_3)\,d\xi$$

$$+ \int_0^t d\tau \int_{\partial D} G_m^{(2)}(t,x,\tau,\xi)\psi_m(\tau,\xi;q_2)\,d_\xi S$$

$$+ \sum_{k=1}^N \Bigg[\int_0^{t_k} d\tau \int_D Z_k^{(1)}(t_k,t,x,\tau,\xi) f_m(\tau,\xi;q_1)\,d\xi$$

$$+ \int_D Z_k^{(1)}(t_k,t,x,0,\xi)\varphi_m(\xi;q_3)\,d\xi$$

$$+ \int_0^{t_k} d\tau \int_{\partial D} Z_k^{(2)}(t_k,t,x,\tau,\xi)\psi_m(\tau,\xi;q_2)\,d_\xi S\Bigg]. \tag{12}$$

Proof. We seek the solution of problem (7)–(9) in the form

$$u_m(t,x) = v_m(t,x) + \int_D G_m^{(1)}(t,x,0,\xi)u_m(0,\xi)\,d\xi, \tag{13}$$

where $v_m(t,x)$ is a solution of problem (11). For $v_m(t,x)$, we get the following representation [2, p. 141]:

$$v_m(t,x) = \int_0^t d\tau \int_D G_m^{(1)}(t,x,\tau,\xi) f_m(\tau,\xi;q_1)\, d\xi + \int_D G_m^{(1)}(t,x,0,\xi)\varphi_m(\xi;q_3)\, d\xi$$

$$+ \int_0^t d\tau \int_{\partial D} G_m^{(2)}(t,x,\tau,\xi)\psi_m(\tau,\xi;q_2)\, d_\xi S. \tag{14}$$

Satisfying the nonlocal condition (8), we get

$$u_m(0,x) + \sum_{k=1}^N \zeta_k(x) \int_D G_m^{(1)}(t_k,x,0,\xi) u_m(0,\xi)\, d\xi$$

$$= -\sum_{k=1}^N \zeta_k(x) v_m(t_k,x) \equiv F_1(x). \tag{15}$$

Since $G_m^{(1)}(t,x,\tau,\xi) \ge 0$ and $\int_D G_m^{(1)}(t,x,\tau,\xi)\, d\xi \le 1$, we find

$$\left| \sum_{j=1}^N \zeta_j(x) \int_D G_m^{(1)}(t_j,x,0,\xi)\, d\xi \right| \le \sum_{j=1}^N |\zeta_j(x)| \int_D G_m^{(1)}(t_j,x,0,\xi)\, d\xi \le \lambda_0.$$

Thus, for the solution of Eq. (15), we get

$$|u_m(0,x)| \le \frac{\lambda_0}{1-\lambda_0} \|F_1;Q\|_0. \tag{16}$$

We solve the integral equation (15) by the method of successive approximations. We represent the solution of the integral equation (15) in the form

$$u_m(0,x) = F_1(x) + \int_D Z_m(x,y) F_1(y)\, dy, \tag{17}$$

where $Z_m(x,y)$ is the resolvent satisfying the integral equation

$$Z_m(x,\xi) + \sum_{k=1}^N \zeta_k(x) G_m^{(1)}(t_k,x,0,\xi)$$

$$= -\int_D \sum_{k=1}^N \zeta_k(x) G_m^{(1)}(t_k,x,0,y) Z_m(y,\xi)\, dy,$$

whence we get the following estimate:

$$\left|\int_D Z_m(x,\xi)\,d\xi\right| \le \frac{\lambda_0}{1-\lambda_0}.$$

Substituting the relation

$$F_1(y) = -\sum_{k=1}^{N}\zeta_k(x)\Bigg[\int_0^{t_k} d\tau \int_D G_m^{(1)}(t,y,\tau,\xi) f_m(\tau,\xi;q_1)\,d\xi + \int_D G_m^{(1)}(t_k,y,0,\xi)\varphi_m(\xi;q_3)\,d\xi + \int_0^{t_k} d\tau \int_{\partial D} G_m^{(2)}(t_k,y,\tau,\xi)\psi_m(\tau,\xi;q_2)\,d_\xi S\Bigg]$$

in equality (17) and changing the order of integration, we obtain

$$u_m(0,x) = \sum_{k=1}^{N}\Bigg[\int_0^{t_k} d\tau \int_D \Gamma_m^{(1)}(t_k,x,\tau,\xi) f_m(\tau,\xi;q_1)\,d\xi + \int_D \Gamma_m^{(1)}(t_k,x,0,\xi)\varphi_m(\xi;q_3)\,d\xi + \int_0^{t_k} d\tau \int_D \Gamma_m^{(2)}(t_k,x,\tau,\xi)\psi_m(\tau,\xi;q_2)\,d_\xi S\Bigg], \tag{18}$$

where

$$\Gamma_m^{(\nu)}(t_k,x,\tau,\xi) = -\zeta_k(x)G_m^{(\nu)}(t_k,x,\tau,\xi) - \int_D \zeta_k(y)Z_m(x,y)G_m^{(\nu)}(t_k,y,\tau,\xi)\,dy, \qquad \nu\in\{1,2\}.$$

Substituting the value of $u_m(0,x)$ in equality (13), applying relation (14) for $v_m(t,x)$, and changing the order of integration, we obtain

$$u_m(t,x) = \int_0^{t} d\tau \int_D G_m^{(1)}(t,x,\tau,\xi) f_m(\tau,\xi;q_1)\,d\xi + \int_D G_m^{(1)}(t,x,0,\xi)\varphi_m(\xi;q_3)\,d\xi + \int_0^{t} d\tau \int_{\partial D} G_m^{(2)}(t,x,\tau,\xi)\psi_m(\tau,\xi;q_2)\,d_\xi S + \sum_{k=1}^{N}\Bigg[\int_0^{t_k} d\tau \int_D Z_k^{(1)}(t_k,t,x,\tau,\xi) f_m(\tau,\xi;q_1)\,d\xi$$

$$+\int_D Z_k^{(1)}(t_k,t,x,0,\xi)\varphi_m(\xi;q_3)d\xi$$

$$\left. +\int_0^{t_k} d\tau \int_{\partial D} Z_k^{(2)}(t_k,t,x,\tau,\xi)\psi_m(\tau,\xi;q_2)d_\xi S \right], \tag{19}$$

where

$$Z_k^{(\nu)}(t_k,t,x,\tau,\xi) = \int_D G_m^{(1)}(t,x,0,y)\Gamma_m^{(\nu)}(t_k,y,\tau,\xi)dy, \quad k\in\{1,\ldots,N\}, \quad \nu\in\{1,2\}.$$

In the domain Q, we consider the following problem:

$$(L_1u_m)(t,x) = f_m(t,x;q_1),$$

$$u_m(0,x) = G_m(x), \tag{20}$$

$$\lim_{x\to z\in\partial D}(B_2u_m-\psi_m)(t,x) = 0,$$

where

$$G_m(x) = \varphi_m(x,q_3)-\sum_{k=1}^N \zeta_k u_m(t_k,x).$$

The solution of the boundary-value problem (20) in the domain Q exists and is unique in the space $C^{2+\alpha}(Q)$ (see [4, p. 90]).

We now estimate the derivatives of the solutions $u_m(t,x)$. In the space $C^{\ell}(Q)$, we introduce the norm $\|u_m;\gamma;\beta;q;Q\|_\ell$. For fixed m_1 and m_2, this norm is equivalent to the Hölder norm, which is expressed in the same way as $\|u;\gamma;\beta;q;Q\|_\ell$ but, instead of the functions $s_1(q^{(1)},t)$ and $s_2(q^{(2)},x)$, it is necessary to take $d_1(q^{(1)},t)$ and $d_2(q^{(2)},x)$, respectively:

$$d_1(q^{(1)},t) = \begin{cases} \max\left(s_1(q^{(1)},t),m_1^{-q^{(1)}}\right), & q^{(1)}\ge 0,\\ \min\left(s_1(q^{(1)},t),m_1^{-q^{(1)}}\right), & q^{(1)}<0,\end{cases}$$

$$d_2(q^{(2)},x) = \begin{cases} \max\left(s_2(q^{(2)},x),m_2^{-q^{(2)}}\right), & q^{(2)}\ge 0,\\ \min\left(s_1(q^{(2)},x),m_2^{-q^{(2)}}\right), & q^{(2)}<0.\end{cases}$$

Theorem 4. *If conditions (1°) and (2°) are satisfied, then the solution of problem (7)–(9) satisfies the following estimate:*

$$\|u_m;\gamma;\beta;0;Q\|_{2+\alpha} \le c(\|\varphi;\tilde{\gamma};\tilde{\beta};0;D\|_{2+\alpha} + \|f;\gamma;\beta;0;Q\|_{2+\alpha} + \|\psi;\gamma;\beta;0;Q\|_{1+\alpha}). \quad (21)$$

The constant c *is independent of* m.

Proof. By using the definition of norm and the interpolation inequalities from [6, 10], we get

$$\|u_m;\gamma;\beta;0;Q\|_{2+\alpha} \le (1+\varepsilon^{\alpha})\langle u_m;\gamma;\beta;0;Q\rangle_{2+\alpha} + c(\varepsilon)\|u_m;Q\|_0,$$

where ε is an arbitrary real number from $(0,1)$. Therefore, it is sufficient to estimate the seminorm $\langle u_m;\gamma;\beta;0;Q\rangle_{2+\alpha}$.

It follows from the definition of the seminorm that the domain Q contains points P_1, P_2, and H_i for which one of the following inequalities is true:

$$\frac{\lambda_0+1}{2}\|u_m;\gamma;\beta;0;Q\|_{2+\alpha} \le E_{\mu}, \qquad \mu\in\{1,2\}, \quad (22)$$

where

$$E_1 \equiv \sum_{2s+|r|=2}\Bigg\{\sum_{\nu=1}^{n} d_1(2s\gamma^{(1)},t^{(2)})d_2(2s\gamma^{(2)},\tilde{x})\prod_{i=1}^{n} d_1(r_i(\gamma^{(1)}-\beta_i^{(1)}),t^{(2)})$$
$$\times d_2(r_i(\gamma^{(2)}-\beta_i^{(2)}),\tilde{x})\left|\partial_t^s\partial_x^r u_m(P_2)-\partial_t^s\partial_x^r u_m(H_\nu)\right|$$
$$\times\left|x_\nu^{(1)}-x_\nu^{(2)}\right|^{-\alpha/2} d_1(\alpha(\gamma^{(1)}-\beta_\nu^{(1)}),t^{(1)})d_2(\alpha(\gamma^{(2)}-\beta_\nu^{(2)}),\tilde{x})\Bigg\},$$

$$E_2 \equiv d_1((2+\alpha)\gamma^{(1)},\tilde{t})d_2((2s+\alpha)\gamma^{(2)},x^{(1)})\prod_{i=1}^{n} d_1(-r_i\beta_i^{(1)},\tilde{t})$$
$$\times d_2(r_i(\gamma^{(2)}-\beta_i^{(2)}),x^{(1)})\left|t^{(1)}-t^{(2)}\right|^{-\alpha/2}\left|\partial_t^s\partial_x^r u_m(P_1)-\partial_t^s\partial_x^r u_m(P_2)\right|,$$
$$2s+|r| = 2.$$

If

$$\left|x_\nu^{(1)}-x_\nu^{(2)}\right| \ge \frac{\varepsilon_1}{4n} d_1(\gamma^{(1)},\tilde{t})\, d_2(\gamma^{(2)}-\beta_\nu^{(2)},\tilde{x}) \equiv T_1$$

and ε_1 is an arbitrary real number from $(0,1)$, then

$$E_1 \le 2\varepsilon_1^{-\alpha} \| u_m; \gamma; \beta; 0; Q \|_2 . \tag{23}$$

If

$$\left| t^{(1)} - t^{(2)} \right| \ge \frac{\varepsilon_1^2}{16} d_1(2\gamma^{(1)}, \tilde{t}) d_2(2\gamma^{(2)}, \tilde{x}) \equiv T_2 ,$$

then

$$E_2 \le 2\varepsilon_1^{-\alpha} \| u_m; \gamma; \beta; 0; Q \|_2 . \tag{24}$$

Applying the interpolation inequalities to (23) and (24), we find

$$E_\mu \le \varepsilon^{\alpha} \| u_m; \gamma; \beta; 0; Q \|_{2+\alpha} + c(\varepsilon) \| u_m; Q \|_0 . \tag{25}$$

Let $\left| x_j^{(1)} - x_j^{(2)} \right| \le T_2$ and let $\left| t^{(1)} - t^{(2)} \right| \le T_1$. We assume that

$$d_1(\gamma^{(1)}, \tilde{t}) \equiv d_1(\gamma^{(1)}, t^{(1)}), \qquad d_2(\gamma^{(2)}, \tilde{x}) \equiv d_2(\gamma^{(2)}, x^{(1)}) .$$

Assume that either $|x_n - \xi_n| \le 2T_2$, $\xi \in \partial D_m$, or $|x - \xi| \le 2T_2 n$.

Consider a sphere $\mathcal{K}(r, P)$ of radius r, $r \ge 4T_2 n$, centered at a point $P \in \Gamma$ and containing the points P_1, H_i, and P_2. By using the restriction imposed on smoothness of the boundary ∂D, we can straighten $\partial D \cap \mathcal{K}(r, P)$ with the help of a bijective transformation $x = \psi(y)$ [10, p. 155], as a result of which the domain

$$\Pi = Q \cap \mathcal{K}(r, P)$$

transforms into the domain Π_1 for the points of which $y_n \ge 0$ and $t \ge 0$.

If we set $u_m(t, x) = \omega_m(t, y)$, $P_1 = R_1$, $H_k = M_k$, $P_2 = R_2$, and $d_2(\gamma^{(2)}, x^{(1)}) = p_2(\gamma^{(2)}, y^{(1)})$ and denote the coefficients of the differential expressions L_1 and B_1 under this transformation by k_{ij}, k_i, k_0, ℓ_i, and ℓ_0, then ω_m is a solution of the problem

$$\left[\partial_t - \sum_{i,j=1}^{n} k_{ij}(R_1) \partial_{y_i} \partial_{y_j} \right] \omega_m = \sum_{i,j=1}^{n} [k_{ij}(t, y) - k_{ij}(R_1)] \partial_{y_i} \partial_{y_j} \omega_m$$

$$+ \sum_{i=1}^{n} k_i(t, y) \partial_{y_i} \omega_m + k_0(t, y) \omega_m + F_m(t, \psi(y)) \equiv F_m^{(0)}(t, y) , \tag{26}$$

$$\omega_m(0, y) = G_m(0, \psi(y)) , \tag{27}$$

$$B_2 \omega_m \big|_{y_n = 0} \equiv \sum_{k=1}^{n} \ell_k(t, R_1) \partial_{y_k} \omega_m \big|_{y_n = 0}$$

$$= \sum_{k=1}^{n}([\ell_k(t,R_1)-\ell_k(t,y)]\partial_{y_k}\omega_m - \ell_0(t,y)\omega_m + \psi_m(t,\psi(y)))\big|_{y_n=0}$$

$$\equiv G_m(t,y)\big|_{y_n=0}. \tag{28}$$

In problem (26)–(28), we perform the change $\omega_m(t,y)=V_m(t,z)$, where

$$z_k = d_1(\beta_k^{(1)},t^{(1)}) \times\ p_2(\beta_k^{(2)},y^{(1)})y_k, \quad k\in\{1,\ldots,n\}.$$

By Π_2 we denote the domain of definition of the functions $V_m(t,z)$. Then the function V_m is a solution of the problem

$$L_3V_m \equiv \left[\partial_t - \sum_{i,j=1}^{n} d_1(\beta_i^{(1)},t^{(1)})d_1(\beta_j^{(1)},t^{(1)})p_2(\beta_i^{(2)},y^{(1)})p_2(\beta_j^{(2)},y^{(1)}) \times k_{ij}(R_1)\partial_{z_i}\partial_{z_j}\right]V_m = F_m^0(t,Z),$$

$$V_m(0,z) = G_m(Z) \equiv \Phi_m(Z),$$

$$B_3V_m\big|_{z_n=0} \equiv \sum_{k=1}^{n} d_1(\beta_k^{(1)},t^{(1)})p_2(\beta_k^{(2)},y^{(1)})\ell_k(t,R_1)\partial_{z_k}V_m\big|_{z_n=0} = R_m(t,Z)\big|_{z_n=0},$$

where

$$Z = (d_1(-\beta_1^{(1)},t^{(1)})p_2(-\beta_1^{(2)},y^{(1)})z_1,\ldots,d_1(-\beta_n^{(1)},t^{(1)})p_2(-\beta_n^{(2)},y^{(1)})z_n).$$

We denote

$$z_i^{(1)}=d_1(\beta_i^{(1)},t^{(1)})p_2(\beta_i^{(2)},y_1)y_i^{(1)}, \quad \Pi_\mu^{(1)} = \{(t,z)\in\Pi_2\,\big|\,|t-t^{(1)}|\le\mu^2T_2, |z_i-z_i^{(1)}|\le\mu\sqrt{T_2},\ i\in\{1,\ldots,n\}\}$$

and take a three times differentiable function $\eta(t,z)$ satisfying the following conditions:

$$\eta(t,z) = \begin{cases} 1, & (t,z)\in\Pi_{1/2}^{(1)}, \quad 0\le\eta(t,z)\le 1, \\ 0, & (t,z)\not\in\Pi_{3/4}^{(1)}, \quad \left|\partial_t^k\partial_z^j\eta(t,z)\right| \le c_{ki}d_1(-(2k+|j|)\gamma^{(1)},t^{(1)})p_2(-(2k+|j|)\gamma^{(2)},y^{(2)}). \end{cases}$$

Then the function $W_m(t,z)=\eta(t,z)V_m(t,z)$ is a solution of the boundary-value problem

$$L_3W_m = \sum\sum_{i,j=1}^{n} d_1(\beta_i^{(1)},t^{(1)})d_1(\beta_j^{(1)},t^{(1)})p_2(\beta_i^{(2)},y^{(1)})p_2(\beta_j^{(2)},y^{(1)})k_{ij}(R_1)$$

$$\times[\partial_{z_i}\eta\partial_{z_j}V_m+\partial_{z_j}\eta\partial_{z_i}V_m]$$

$$+V_m\Bigg[\sum_{i,j=1}^{n} d_1(\beta_i^{(1)},t^{(1)})d_1(\beta_j^{(1)},t^{(1)})p_2(\beta_i^{(2)},y^{(1)})p_2(\beta_j^{(2)},y^{(1)})$$

$$\times k_{ij}(R_1)\partial_{z_i}\partial_{z_j}\eta-\partial_t\eta\Bigg]+F_m^{(0)}\eta \equiv F_m^{(1)}(t,z), \tag{29}$$

$$W_m(0,x) = \eta(0,z)\Phi_m(Z) \equiv \Phi_m^{(1)}(z), \tag{30}$$

$$B_3W_m|_{z_n=0} = \left[\sum_{k=1}^{n} d_1(\beta_k^{(1)},t^{(1)})p_2(\beta_k^{(2)},y^{(1)})V_m\ell_k(t,R_1)\partial_{z_k}\eta - R_m(t,Z)\eta\right]\Bigg|_{z_n=0} \equiv R_m^{(1)}. \tag{31}$$

According to the imposed conditions, the coefficients in Eq. (29) and in the boundary condition (31) are bounded by constants independent of the point R_1. Therefore, by Theorem 6.2 in [2, p. 368], for any points $\{M_1,M_1\}\subset\Pi_{1/2}^{(1)}$, we arrive at the inequality

$$d^{-\alpha}(M_1,M_2)\left|\partial_t^k\partial_z^jV_m(M_1)-\partial_t^k\partial_z^jV_m(M_2)\right|$$

$$\le c\left(\left\|F_m^{(1)}\right\|_{C^\alpha(\Pi_{3/4}^{(1)})}+\left\|\Phi_m^{(1)}\right\|_{C^{2+\alpha}(\Pi_{3/4}^{(1)}\cap\{t=t_k\})}+\left\|G_1\right\|_{C^{1+\alpha}(\Pi_{3/4}^{(1)}\cap\{(t,z)\in\Pi_{3/4}^{(1)}|z_n=0\})}\right), \tag{32}$$

where $2k+|j|=2$ and $d(M_1,M_2)$ is the parabolic distance between M_1 and M_2.

In view of the properties of the function $\eta(t,z)$, we obtain

$$\left\|F_m^{(1)}\right\|_{C^\alpha(\Pi_{3/4}^{(1)})} \le cd_1(-(2+\alpha)\gamma^{(1)},t^{(1)})p_2(-(2+\alpha)\gamma^{(2)},y^{(1)})$$

$$\times(\left\|F_m;\gamma;0,2\gamma;\Pi_{3/4}^{(1)}\right\|_\alpha+\left\|V_m;\Pi_{3/4}^{(1)}\right\|_0+\left\|V_m;\gamma;0,0;\Pi_{3/4}^{(1)}\right\|_2),$$

$$\left\|\Phi_m^{(1)}\right\|_{C^{2+\alpha}(\Pi_{3/4}^{(1)}\cap(t=0))} \le cd_1(-(2+\alpha)\gamma^{(1)},t^{(1)})p_2(-(2+\alpha)\gamma^{(2)},y^{(1)})$$

$$\times\left\|\Phi_m;\tilde{\gamma};0;0;\Pi_{3/4}^{(1)}\cap\{t=0\}_{2+\alpha}\right\|, \tag{33}$$

$$\left\|R_m^{(1)}\right\|_{C^{1+\alpha}(\Pi_{3/4}^{(1)}\cap\{(t,z)\in\Pi_{3/4}^{(1)}|z_n=0\})} \le cd_1(-(2+\alpha)\gamma^{(1)},t^{(1)})$$

$$\times p_2(-(2+\alpha)\gamma^{(2)},y^{(1)})(\left\|R_m;\gamma,0;\gamma;\Pi_{3/4}^{(1)}\right\|_{1+\alpha}$$

$$+\left\|V_m;\gamma,0;0;\Pi_{3/4}^{(1)}\right\|_2+\left\|V_m;\Pi_{3/4}^{(1)}\right\|_0).$$

Substituting (33) in (32) and returning to the variables (t,y), we find

$$E_r \le c(\|F_m;\gamma,\beta;2\gamma;\Pi_2\|_\alpha+\left\|\Phi_m;\tilde{\gamma},\tilde{\beta};0;\Pi_2\cap\{t=0\}\right\|_{2+\alpha}$$

$$+\left\|G_m;\gamma,\beta;\gamma;\Pi_{3/4}^{(1)}\right\|_{1+\alpha}+c_1\|V_m;\gamma,\beta;0;\Pi_2\|_2+\|V_m;\Pi_2\|_0),\quad r\in\{1,2\}. \tag{34}$$

Taking into account the definition of the space $H^{2+\alpha}(\gamma,\beta;0;Q)$ and conditions (1°) and (2°), we get

$$E_\mu \le c(n^2\rho^\alpha+\varepsilon^\alpha(n+2))\|u_m;\gamma,\beta;0;Q\|_{2+\alpha}+c_1(\|f_m;\gamma,\beta;0;Q\|_{2+\alpha}$$

$$+\lambda_0\|u_m;\gamma;\beta;0;Q\|_{2+\alpha}+\left\|\varphi_m;\tilde{\gamma};\tilde{\beta};0;Q\cap(t=0)\right\|_{2+\alpha}$$

$$+\|\psi_m;\gamma,\beta;\gamma;Q\|_{1+\alpha}+\|u_m;Q\|_0),\quad \mu\in\{1,2\}, \tag{35}$$

ε and ρ are arbitrary numbers, $\rho\in(0,1)$, and $\varepsilon\in(0,1)$.

Let $|x-\xi|\ge 2T_2 n$. Consider a problem

$$\left[\partial_t-\sum_{i,j=1}^n a_{ij}(P)\partial_{x_i}\partial_{x_j}\right]u_m \equiv \sum_{i,j=1}^n [a_{ij}(t,x)-a_{ij}(P_1)]\partial_{x_i}\partial_{x_j}u_m$$

$$+\sum_{i=1}^n(a_i(t,x)\partial_{x_i}u_m+a_0(t,x))u_m+f_m \equiv F_m^{(2)}(t,z), \tag{36}$$

$$u_m(0,x) = G_m(x). \tag{37}$$

Let $\Pi_1^{(2)}\in Q$, let $\Pi_1^{(2)}$ be a cube centered at the point P_1, and let

$$\Pi_\rho^{(2)}=\{(t,x)\in Q^{(k)}\,\big|\,|t-t^{(1)}|\le 16^{-1}\mu^2T_1,|x_i-x_i^{(1)}|\le 4\mu^{-1}T_2,i\in\{1,2,\ldots,n\}\}.$$

In problem (36), (37), we perform the change of variables

$$u_m(t,x)=\omega_m(t,y),\quad x_i=d_1(\beta_i^{(1)},t^{(1)})d_2(\beta_i^{(2)},x^{(1)})y_i,\quad i\in\{1,2,\ldots,n\}.$$

By $\Pi^{(3)}$ we denote the domain of definition of the functions $\omega_m(t,y)$. Then $\omega_m(t,y)$ is a solution of the problem

$$L_3\omega_m \equiv \left[\partial_t - \sum_{i,j=1}^{n} a_{ij}(P_1) d_1(\beta_i^{(1)}, t^{(1)}) d_1(\beta_j^{(1)}, t^{(1)}) \right.$$

$$\left. \times d_2(\beta_i^{(2)}, x^{(1)}) d_2(\beta_j^{(2)}, x^{(1)}) \partial_{x_i} \partial_{x_j} \right] \omega_m = F_m^{(2)}(t, Y), \tag{38}$$

$$\omega_m(0, y) = G_m(Y), \tag{39}$$

where

$$Y = \left(d_1(-\beta_1^{(1)}, t^{(1)}) d_2(-\beta_1^{(2)}, x^{(1)}) y_1, \ldots, d_1(-\beta_n^{(1)}, t^{(1)}) d_2(-\beta_n^{(2)}, x^{(1)}) y_n\right).$$

We denote

$$y_i^{(1)} = d_1(\beta_i^{(1)}, t^{(1)}) d_2(\beta_i^{(2)}, x^{(1)}) x_i^{(1)}$$

and

$$\Pi_\mu^{(3)} = \{(t, y) \in \Pi^{(3)} \mid |t - t^{(1)}| \le 16^{-1}\mu^2 T_1, |x_i^{(1)} - x_i| \le 4^{-1}\mu\sqrt{T_1}, i \in \{1, \ldots, n\}\}$$

and take a trice differentiable function $\eta_1(t, y)$

$$\eta_1(t, y) = \begin{cases} 1, & (t, y) \in \Pi_{1/2}^{(3)}, \quad 0 \le \eta_1(t, y) \le 1, \\ 0, & (t, y) \not\subset \Pi_{3/4}^{(3)}, \quad \left|\partial_t^k \partial_x^j \eta_1(t, y)\right| \le c_{kj} d_1(-(2k + |j|)\gamma^{(1)}, t^{(1)}) d_2(-(2k + |j|)\gamma^{(2)}, x^{(1)}). \end{cases}$$

Then the function $V_m^{(1)}(t, y) = \omega_m(t, y)\eta_1(t, y)$ is a solution of the Cauchy problem

$$L_3 V_m^{(1)} = \sum_{i,j=1}^{n} a_{ij}(P_1) d_1(\beta_i^{(1)}, t^{(1)}) d_1(\beta_j^{(1)}, t^{(1)}) d_2(\beta_i^{(2)}, x^{(1)}) d_2(\beta_j^{(2)}, x^{(1)})$$

$$\times [\partial_{y_i} \eta_1 \partial_{y_j} \omega_m + \partial_{y_j} \eta_1 \partial_{y_i} \omega_m]$$

$$+ \omega_m \left[\sum_{i,j=1}^{n} a_{ij}(P_1) d_1(\beta_i^{(1)}, t^{(1)}) d_1(\beta_j^{(1)}, t^{(1)}) d_2(\beta_i^{(2)}, x^{(1)}) \right.$$

$$\left. \times d_2(\beta_j^{(2)}, x^{(1)}) \partial_{y_i} \partial_{y_j} \eta_1 - \partial_t \eta_1 \right] + F_m^{(2)} \eta_1 \equiv F_m^{(3)}, \tag{40}$$

$$V_m^{(1)}(0,y) = G_m \eta_1(0,y) = R_m^{(2)}. \tag{41}$$

According to Theorem 5.1 in [2, p. 364], for any points $\{M_1, M_2\} \subset \Pi_{1/2}^{(3)}$, the inequality

$$d^{-\alpha}(M_1,M_2)\left|\partial_t^k \partial_x^j \omega_m(M_1) - \partial_t^k \partial_x^j \omega_m(M_2)\right|$$

$$\le c\left(\left\|F_m^{(3)}\right\|_{C^{\alpha}(V_{3/4}^{(2)})} + \left\|R_m^{(2)}\right\|_{C^{2+\alpha}(V_{3/4}^{(1)}\cap\{t=0\})}\right), \qquad 2k+|j| = 2,$$

is true.

In view of the properties of the function $\eta_1(t,y)$, the definition of the space $H^{2+\alpha}(\gamma;\beta;0;Q)$, and restrictions (1°) and (2°), we arrive at the inequality

$$E_\mu \le c(n^2\rho^\alpha + \varepsilon^\alpha(n+2))\|u_m;\gamma;\beta;0;Q\|_{2+\alpha} + \lambda_0\|u_m;\gamma;\beta;0;Q\|_{2+\alpha}$$

$$+ c_1(\|\varphi_m;\tilde{\gamma};\tilde{\beta};0;D\|_{2+\alpha} + \|f_m;\gamma;\beta;0;Q\|_\alpha). \tag{42}$$

Combining inequalities (22), (25), (35), and (42) and choosing sufficiently small ε, we obtain the inequalities

$$\|u_m;\gamma;\beta;0;Q\|_{2+\alpha} \le c(\|f_m;\gamma;\beta;0;Q\|_\alpha + \|\varphi_m;\tilde{\gamma};\tilde{\beta};0;D\|_{2+\alpha}$$

$$+ \|\psi_m;\gamma;\beta;0;Q\|_{1+\alpha}). \tag{43}$$

Since

$$\|f_m;\gamma;\beta;0;Q\|_\alpha \le c\|f;\gamma;\beta;0;Q\|_\alpha,$$

$$\|\varphi_m;\tilde{\gamma};\tilde{\beta};0;D\|_{2+\alpha} \le c\|\varphi_k;\tilde{\gamma};\tilde{\beta};0;D\|_{2+\alpha}, \tag{44}$$

$$\|\psi_m;\gamma;\beta;\delta;Q\|_{1+\alpha} \le c\|\psi;\gamma;\beta;\delta;Q\|_{1+\alpha},$$

substituting (37) in (36), we get inequality (21).

Proof of Theorem 1. The right-hand side of inequality (21) does not depend on m_1 and m_2 and the sequences

$$\{u_m^{(0)}\} \equiv \{u_m(P)\}, \qquad P(t,x) \in Q,$$

$$\{u_m^{(1)}\} \equiv \{d_1(\gamma^{(1)} - \beta_i^{(1)}, t)\, d_2(\gamma^{(2)} - \beta_i^{(2)}, x)\, \partial_{x_i} u_m\},$$

$$\{u_m^{(2)}\} \equiv \{d_1(2\gamma^{(1)},t)d_2(2\gamma^{(2)},x)\partial_t u_m\},$$

$$\{u_m^{(3)}\} \equiv \{d_1(\gamma^{(1)}-\beta_i^{(1)},t)d_1(\gamma^{(1)}-\beta_j^{(1)},t)d_2(\gamma^{(2)}-\beta_i^{(2)},x)\ d_2(\gamma^{(2)}-\beta_j^{(2)},x)\partial_{x_i}\partial_{x_j}u_m\}$$

are uniformly bounded and equicontinuous in the domain Q. According to the Arzelà theorem, there exist subsequences $\{u_{m(\ell)}^{(\mu)}\}$ uniformly convergent in Q to $\{u^{(\mu)}\}$, $\mu \in \{0,1,2,3\}$. Passing to the limit as $m(\ell) \to \infty$ in problem (7)–(9), we conclude that $u(t,x) = u_0^{(0)}$ is a unique solution of problem (1)–(3), $u \in H^{2+\alpha}(\gamma;\beta;0;Q)$, and estimate (6) is true.

3. Problem of Optimal Control

In the domain Q, we consider problem (1)–(4). Assume that both conditions (1°), (2°) and the following conditions are satisfied:

(3°) the functions $f(t,x,q_1) = r^{(0)}(t)f^{(0)}(x,q_1(x))$, $\psi(t,x,q_2) = r^{(1)}(t)\psi^{(0)}(x,q_2(x))$, $F_1(t,x;u;q_1)$, $F_2(t,x;u;q_2)$, and $F_3(x,u(t_1,x,q),u(t_2,x,q),\ldots,u(t_N,x,q),q_3)$ have the second-order derivatives with respect to the variables (u,q_1,q_2,q_3), which belong, as functions of the variables (t,x) and x, to the spaces $C^{\alpha}(Q)$, $C^{1+\alpha}(\Gamma)$, and $C^{2+\alpha}(D)$, respectively.

To solve problem (1)–(4), we construct a sequence of solutions of the problems whose limit value is a solution of problem (1)–(4).

In the domain Q, we consider the problem of finding the functions (u_m, q) for which the functional

$$I(q) = \int_0^T dt \int_D F_1(t,x;u_m(t,x,q),q_1)\,dx + \int_0^T dt \int_D F_2(t,x;u_m(t,x,q),q_2)\,d_xS$$

$$+ \int_D F_3(t,x;u_m(t_1,x,q),\ldots,u_m(t_N,x,q),q_3(x))\,dx \tag{45}$$

attains its minimum value in the class of functions $q \in V$, where u_m satisfies Eq. (7) for

$$f_m(t,x,q_1) = r_m^{(0)}(t)f_m^{(0)}(x,q_1(x)),$$

the multipoint condition (8) with respect to the time variable, and the boundary condition (9) on the lateral surface Γ for

$$\psi_m(t,x;q_2) \equiv r_m^{(1)}(t)\psi_m^{(0)}(x,q_2(x)).$$

Denote

$$\boldsymbol{\omega} = (u_m(t_1,x,q),u_m(t_2,x,q),\ldots,u_m(t_N,x,q),q_3) = (\omega_1,\omega_2,\ldots,\omega_{N+1}),$$

$$\lambda_1(\xi) = \int_0^T r_m^{(0)}(\tau)d\tau\int_\tau^T dt\int_D \frac{\partial F_1(t,x;u_m(t,x,q),q_1)}{\partial u_m}G_m^{(1)}(t,x,\tau,\xi)dx$$

$$+\sum_{k=1}^N\int_0^{t_k} r_m^{(0)}(\tau)d\tau\int_\tau^T dt\int_D \frac{\partial F_1(t,x;u_m(t,x,q),q_1)}{\partial u_m}Z_k^{(1)}(t_k,t,x,\tau,\xi)dx$$

$$+\int_0^T r_m^{(0)}(\tau)d\tau\int_\tau^T dt\int_{\partial D} \frac{\partial F_2(t,x;u_m(t,x,q),q_1)}{\partial u_m}G_m^{(2)}(t,x,\tau,\xi)d_xS$$

$$+\sum_{k=1}^N\int_0^{t_k} r_m^{(0)}(\tau)d\tau\int_\tau^T dt\int_{\partial D} \frac{\partial F_2(t,x;u_m(t,x,q),q_1)}{\partial u_m}Z_k^{(2)}(t_k,t,x,\tau,\xi)d_xS$$

$$+\sum_{j=1}^N\left[\int_0^{t_j} r_m^{(0)}(\tau)d\tau\int_D \frac{\partial F_3(x;\omega)}{\partial \omega_j}G_m^{(1)}(t_j,x,\tau,\xi)dx\right.$$

$$\left.+\sum_{k=1}^N\int_0^{t_k} r_m^{(0)}(\tau)d\tau\int_D \frac{\partial F_3(x;\omega)}{\partial \omega_j}Z_k^{(1)}(t_k,t_j,x,\tau,\xi)dx\right],$$

$$\lambda_2(\xi) = \int_0^T r_m^{(1)}(\tau)d\tau\int_\tau^T dt\int_D \frac{\partial F_1(t,x;u_m(t,x,q),q_1)}{\partial u_m}G_m^{(1)}(t,x,\tau,\xi)dx$$

$$+\sum_{k=1}^N\int_0^{t_k} r_m^{(1)}(\tau)d\tau\int_\tau^T dt\int_D \frac{\partial F_1(t,x;u_m(t,x,q),q_1)}{\partial u_m}Z_k^{(1)}(t_k,t,x,\tau,\xi)dx$$

$$+\int_0^T r_m^{(1)}(\tau)d\tau\int_\tau^T dt\int_{\partial D} \frac{\partial F_2(t,x;u_m(t,x,q),q_2)}{\partial u_m}G_m^{(2)}(t,x,\tau,\xi)d_xS$$

$$+\sum_{k=1}^N\int_0^{t_k} r_m^{(1)}(\tau)d\tau\int_\tau^T dt\int_{\partial D} \frac{\partial F_2(t,x;u_m(t,x,q),q_2)}{\partial u_m}Z_k^{(2)}(t_k,t,x,\tau,\xi)d_xS$$

$$+\sum_{j=1}^N\left[\int_0^{t_j} r_m^{(1)}(\tau)d\tau\int_{\partial D} \frac{\partial F_3(x;\omega)}{\partial \omega_j}G_m^{(2)}(t_j,x,\tau,\xi)d_xS\right.$$

$$\left.+\sum_{k=1}^N\int_0^{t_k} r_m^{(1)}(\tau)d\tau\int_{\partial D} \frac{\partial F_3(x;\omega)}{\partial \omega_j}Z_k^{(2)}(t_k,t_j,x,\tau,\xi)d_xS\right],$$

$$\lambda_3(\xi) = \int_0^T dt \int_D \frac{\partial F_1(t,x;u_m(t,x,q),q_1)}{\partial u_m} G_m^{(1)}(t,x,0,\xi)dx$$

$$+\sum_{k=1}^N \int_0^T dt \int_D \frac{\partial F_1(t,x;u_m(t,x,q),q_1)}{\partial u_m} Z_k^{(1)}(t_k,t,x,0,\xi)dx$$

$$+\int_D \left[\sum_{j=1}^N \frac{\partial F_3(x;\boldsymbol{\omega})}{\partial \omega_j} G_m^{(1)}(t_j,x,0,\xi) \right.$$

$$\left. +\sum_{k=1}^N \frac{\partial F_3(x;\boldsymbol{\omega})}{\partial \omega_j} Z_k^{(1)}(t_k,t_j,x,0,\xi) \right] dx$$

$$+\int_0^T dt \int_{\partial D} \frac{\partial F_2(t,x;u_m(t,x,q),q_2)}{\partial u_m} G_m^{(2)}(t,x,0,\xi)d_xS$$

$$+\sum_{k=1}^N \int_0^{t_k} dt \int_{\partial D} \frac{\partial F_2(t,x;u_m(t,x,q),q_2)}{\partial u_m} Z_k^{(2)}(t_k,t,x,0,\xi)d_xS,$$

$$H_1(\xi,u_m,\lambda_1,q_1) \equiv \lambda_1(\xi) f_m^{(0)}(\xi,q_1(\xi)) + \int_0^T F_1(t,\xi;u_m,q_1)dt,$$

$$H_2(\xi,u_m,\lambda_2,q_2) \equiv \lambda_2(\xi) \psi_m^{(0)}(\xi,q_2(\xi)) + \int_0^T F_2(t,\xi;u_m,q_2)dt,$$

$$H_3(\xi,u_m,\lambda_3,q_3) \equiv \lambda_3(\xi) \varphi_m(\xi,q_3(\xi)) + F_3(\xi;\boldsymbol{\omega}),$$

$q^{(0)} = (q_1^{(0)}, q_2^{(0)}, q_3^{(0)})$ is the optimal control, and $u_m(t,x,q^{(0)})$ is the optimal solution of problem (7)–(9).

The following theorem is true:

Theorem 5. *If*

$$\partial_{q_i} H_i(\xi,u_m,\lambda_i,q_i) > 0,$$

then the optimal control is $q_i^{(0)} = V_{i1}$, $i \in \{1,2,3\}$. *If*

$$\partial_{q_i} H_i(\xi,u_m,\lambda_i,q_i) < 0,$$

then the optimal control is $q_i^{(0)} = V_{i2}$, $i \in \{1,2,3\}$.

Proof. Consider, e.g., the case $i = 3$. Let Δq_3 be an arbitrary increment of the control $q_3(x) \in V$, $\Delta q_3 > 0$, $q_3 + \Delta q_3 \in V$. By $\Delta_{q_3} u_m$ we denote the corresponding increment of the function $u_m(t,x;q)$. Then $\Delta_{q_3} u_m$ is a solution of the boundary-value problem

$$(L_1 \Delta_{q_3} u_m)(t,x) = 0, \qquad \lim_{x \to z \in \partial D} (B_2 \Delta_{q_3} u_m)(t,x) = 0,$$
$$(B \Delta_{q_3} u_m)(x) = \varphi_m(x; q_3 + \Delta q_3) - \varphi_m(x; q_3) \equiv \Delta_{q_3} \varphi_m(x; q_3) \tag{46}$$

in the domain Q.

We represent a partial increment of the functional $I(q)$ by the Taylor formula as follows:

$$\Delta_{q_3} I = \int_0^T dt \int_D \partial_{u_m} F_1(t,x;u_m,q_1) \Delta_{q_3} u_m \, dx$$
$$+ \int_0^T dt \int_{\partial D} \partial_{u_m} F_2(t,x;u_m,q_2) \Delta_{q_3} u_m \, d_x S$$
$$+ \sum_{k=1}^N \int_D \partial_{\omega_k} F_3(x;\boldsymbol{\omega}) \Delta_{q_3} \omega_k \, dx + \int_D \partial_{q_3} F_3(x;\boldsymbol{\omega}) \Delta q_3 \, dx$$
$$+ \int_0^T dt \int_D O(|\Delta_{q_3} u_m|^2) dx + \int_D \Big[O(|\Delta_{q_3}|^2)$$
$$+ \sum_{k=1}^N O(|\Delta_{q_3} \omega_k|^2) \Big] dx + \int_0^T dt \int_{\partial D} O(|\Delta_{q_3} u_m|^2) d_x S. \tag{47}$$

Since $\Delta_{q_3} u_m$ is a solution of problem (46), by using formula (12), we obtain

$$\Delta_{q_3} u = \int_D G_m^{(1)}(t,x,0,\xi) \Delta_{q_3} \varphi_m(\xi; q_3) d\xi$$
$$+ \sum_{k=1}^N \int_D Z_k^{(1)}(t_k, t, x, 0, \xi) \Delta_{q_3} \varphi_m(\xi; q_3) d\xi. \tag{48}$$

Substituting (48) in (47) and changing the order of integration, we find

$$\Delta_{q_3} I = \int_D \Big[\partial_{q_3} H_3(\xi, u_m, \lambda_3, q_3) \Delta q_3 + O(|\Delta_{q_3} \Delta u_m|^2) + O(|\Delta q_3|^2) \Big] dx. \tag{49}$$

If $q_3 = V_{31}(x)$ and $\partial_{q_3} H_3 > 0$, then, for sufficiently small Δq_3, we get $\Delta_{q_3} I > 0$. If $q_3 = V_{32}(x)$ and $\partial_{q_3} H_3 < 0$, then, for sufficiently small Δq_3, we have $\Delta_{q_3} I > 0$. If $H_3(\xi, u_m, \lambda_3, q_3)$ is not monotone as a function of the argument q_3, then $\partial_{q_3} H_3(\xi, u_m, \lambda_3, q_3)$ is an alternating function: $\partial_{q_3} H_3(\xi, u_m, \lambda_3, q_3) > 0$ in the domain $D^+ \subset D$ and $\partial_{q_3} H_3(\xi, u_m, \lambda_3, q_3) < 0$ in the domain $D^- = D \setminus D^+$.

By the mean-value theorem, we find

$$\Delta_{q_3} I = \partial_{q_3} H_3(\xi^+, u_m^+, \lambda_3^+, q_3^+) \int_{D^+} \Delta q_3 \, dx$$

$$- \left| \partial_{q_3} H_3(\xi^-, u_m^-, \lambda_3^-, q_3^-) \right| \int_{D^-} \Delta q_3 \, dx$$

$$+ \int_D \left[O(|\Delta_{q_3} u_m|^2) + O(|\Delta q_3|^2) \right] dx.$$

For sufficiently small Δq_3, the sign of $\Delta_{q_3} I$ is determined by the first terms depending on the quantities $\operatorname{mes} D^+$, $\operatorname{mes} D^-$, and Δq_3. Thus, the functional $I(q)$ for the control q_3 does not attain its minimum value. Similar reasoning should also be used in the case where $\Delta q_3 < 0$.

In proving the theorem in the cases where $i \in \{1,2\}$, it is necessary to use the scheme applied in the case $i = 3$.

Assume that the conditions of Theorem 5 are not satisfied. Then the following theorem is valid:

Theorem 6. *In order that the control* $q^{(0)} = \{q_1^{(0)}, q_2^{(0)}, q_3^{(0)}\}$ *be optimal, it is necessary and sufficient that the following conditions be satisfied*:

(i) the functions $H_i(\xi, u_m, \lambda_i, q_i)$ *take their minimum value as functions of the argument* q_i *at the point* $q_i^{(0)}$, $i \in \{1,2,3\}$;

(ii) for any vector $(e_k^{(1)}, e_k^{(2)}) \neq 0$, *the inequality*

$$\partial_{u_m}^2 F_k(t, x; u_m; q_k^{(0)}) (e_k^{(1)})^2 + 2 \partial_{q_k} \partial_{u_m} F_k(t, x; u_m; q_k^{(0)}) e_k^{(1)} e_k^{(2)}$$

$$+ \partial_{q_k}^2 F_k(t, x; u_m; q_k^{(0)}) (e_k^{(2)})^2 > 0, \qquad k \in \{1,2\},$$

is true;

(iii) for any vector $(e_1, e_2, \ldots, e_{N+1}) \neq 0$, *the inequality*

$$\sum_{i,j=1}^{N+1} \partial_{\omega_i \omega_j}^2 F_3(x; \boldsymbol{\omega}) e_i e_j > 0$$

holds.

Proof. The proof of Theorem 6 is carried out by the methods proposed in [7, 9]. Passing to the limit in problem (7)–(9), (45) as $m_1 \to \infty$ and $m_2 \to \infty$, we arrive at the optimal solution of problem (1)–(4).

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