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INTERNAL AND STARTUP CONTROLS OF THE SOLUTIONS OF BOUNDARY-VALUE PROBLEM FOR PARABOLIC EQUATIONS WITH DEGENERATIONS

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For a second-order parabolic equation with degenerations, we construct the solution of the problem of optimal control for systems described by the first boundary-value problem with internal and startup controls. The coefficients of parabolic equation have power singularities of any order in time and space variables on a certain set of points.

Keywords: parabolic boundary-value problem, power singularities, Hölder spaces, Green function, optimal solution.

In the contemporary applied investigations of partial differential equations, the researchers often encounter the problems with nonclassical conditions in the equations, boundary operators, and various singularities and degenerations. These problems are of especial interest for parabolic equations and systems of equations, which describe the processes of diffusion of liquids and gases, pressure, the process of heat conduction, and other processes running in bodies of complex configurations.

Depending on the structure of medium, the processes of diffusion, heat conduction, and thermoelasticity are modeled by parabolic differential equations. In this case, both equations and boundary conditions have nonclassical restrictions and are characterized by various degenerations, random perturbations, and impulsive actions.

Thus, the problem of establishing correct solvability of the boundary-value problems, finding the representations of their solutions, and determination of the properties of these solutions seems to be among the most important problems of mathematical physics, which is of high interest for the researchers.

The comprehensive presentation of the theory of boundary-value problems for linear differential equations with degenerations can be found in the works by O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Ural'tseva [4], E. M. Landis [5], A. Friedman [11], S. D. Eidelman, S. D. Ivasyshen, and A. N. Kochubei [12], H. Lange and H. Teismann [13], M. I. Matiichuk [6], B. Yo. Ptashnyk, V. S. Il'kiv, I. Ya. Kmit', and V. M. Polishchuk [7], I. D. Pukal's'kyi [8, 9], etc.

In the present paper, we study a parabolic boundary-value problem for a linear differential equation with power singularities in coordinate planes of any order. The obtained results are used for the investigation of the problems of optimal control with internal and startup controls. As the quality criterion, we choose the sum of volume and surface integrals [9]. The present paper can be regarded as a continuation of our investigations of other boundary-value problems for parabolic equations with degenerations [1–3, 10].

1. Statement of the Problem and Main Restrictions

In a domain $Q=[0,T)\times D$, we consider the problem of finding functions (u,p,q) on which the functional

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$$I(p,q) = \int_0^T dt \int_D F_1(t,x;u(t,x;p(t,x),q(x)),p(t,x))dx$$

$$+ \int_D F_2(x;u(T,x;p(T,x),q(x)),q(x))dx, \qquad (1)$$

reaches its minimum in the class of functions $(p,q) \in V = \{p \in C^{\alpha}(Q): p_1(t,x) \le p \le p_2(t,x);\quad q(x) \in C^{2+\alpha}(D), q_1(x) \le q(x) \le q_2(x)\}$ for which the function $u(t,x;p(t,x),q(x))$ is a solution of the boundary-value problem

$$(Lu)(t,x) \equiv \left[\partial_t - \sum_{i,j=1}^{n} A_{ij}(t,x)\partial_{x_i}\partial_{x_j} - \sum_{i=1}^{n} A_i(t,x)\partial_{x_i} - A_0(t,x)\right]u = f(t,x;p(t,x)), \qquad (2)$$

$$u(0,x;p(0,x),q(x)) = \varphi(x;q(x)), \qquad (3)$$

$$\lim_{x \to z \in \partial D} \left[u(t,x;p(t,x),q(t)) - \psi(t,x)\right] = 0. \qquad (4)$$

The order of singularity of the differential expression L at an arbitrary point $P(t,x) \in Q$ is characterized by the functions $s_1(\beta_i^{(1)},t)$ and $s_2(\beta_i^{(2)},x_i)$:

$$s_1(\beta_i^{(1)},t) = \begin{cases} |t-t_0|^{\beta_i^{(1)}}, & |t-t_0| \le 1, \\ 1, & |t-t_0| > 1, \end{cases} \qquad s_2(\beta_i^{(2)}, x_i) = \begin{cases} x_i^{\beta_i^{(2)}}, & 0 \le x_i \le 1, \\ 1, & x_i > 1, \end{cases}$$

where $\beta_i = (\beta_i^{(1)},\beta_i^{(2)})$, $i \in \{1,\ldots,n\}$, $\beta^{(\nu)} = (\beta_1^{(\nu)},\ldots,\beta_n^{(\nu)})$, $\beta_i^{(\nu)}$ are real numbers, and $\beta_i^{(\nu)} \in (-\infty,\infty)$, $\nu \in \{1,2\}$.

By ℓ, $\gamma^{(1)}$, $\gamma^{(2)}$, and q we denote real numbers; $\ell > 0$, $[\ell]$ is the integer part of a number ℓ, $\{\ell\} = \ell - [\ell]$, $\gamma^{(1)} \ge 0$, $\gamma^{(2)} \ge 0$, $q \ge 0$. By $(x_1^{(1)},\ldots,x_\nu^{(1)},\ldots,x_n^{(1)})$ we denote the coordinates of a point $x^{(1)}$ in the domain $\bar{D} = D \cup \partial D$, where ∂D is the boundary of domain D and by $(x_1^{(1)},\ldots,x_{\nu-1}^{(1)},x_\nu^{(2)},x_{\nu+1}^{(1)},\ldots,x_n^{(1)})$ we denote the coordinates of a point $x^{(2)}$ in the domain $\bar{D}$, $\nu \in \{1,2\}$. Here, $P_1(t^{(1)},x^{(1)})$, $H_i(t^{(1)},x^{(2)})$, and $P_2(t^{(2)},x^{(2)})$ are arbitrary points of the domain $\bar{Q}$, $i \in \{1,\ldots,n\}$.

We now define the spaces of functions in which we study problem (2)–(4).

Let $C^{\ell}(\gamma;\beta;q;Q)$ be the set of functions $u(t,x)$ from the space $L_1(Q)$, $(t,x) \in Q$, with partial derivatives at $t \ne t_0$ of the form $\partial_t^j \partial_x^k u$ (here, $2j + |k| \le [\ell]$, $|k| = k_1 + \ldots + k_n$, $\partial_x^k = \partial_{x_1}^{k_1}\partial_{x_2}^{k_2}\ldots\partial_{x_n}^{k_n}$) for which the norm

$$\|u;\gamma;\beta;q;Q\|_{\ell} = \|u;\gamma;\beta;q;Q\|_{[\ell]} + \langle u;\gamma;\beta;q;Q\rangle_{\ell}$$

is finite. Thus in particular,

$$\|u;\gamma;\beta;q;Q\|_0 = \sup_{\bar{Q}}|u| \equiv \|u;Q\|_0,$$

$$\|u;\gamma;\beta;q;Q\|_{[\ell]} = \sum_{2j+|k|\le[\ell]} \sup_{P\in Q}\Bigg[S(q+(2j+|k|)\gamma,P)\prod_{i=1}^{n} s_1(-k_i\beta_i^{(1)},t)$$

$$\times s_2(-k_i\beta_i^{(2)},x_i)\left|\partial_t^j\partial_x^k u(P)\right|\Bigg],$$

$$\langle u;\gamma;\beta;q;Q\rangle_\ell = \sum_{2j+|k|=[\ell]}\sum_{i=1}^{n}\sup_{(P_1,H_i)\subset\bar{Q}}\left|x_i^{(1)}-x_i^{(2)}\right|^{-\{\ell\}}$$

$$\times\left|\partial_t^j\partial_x^k u(P_1)-\partial_t^j\partial_x^k u(H_i)\right|S((\ell+q)\gamma;\tilde{P})s_1(-\{\ell\}\beta_i^{(1)},t^{(1)})$$

$$\times s_2(-\{\ell\}\beta_i^{(2)},\tilde{x}_i)\prod_{m=1}^{n} s_2(-k_m\beta_m^{(2)},\tilde{x}_m)s_1(-k_m\beta_m^{(1)},t)$$

$$+\sum_{2j+|k|=[\ell]}\sum_{i=1}^{n}\sup_{(P_2,H_i)\subset\bar{Q}}\left|t^{(1)}-t^{(2)}\right|^{-\{\ell\}/2}\left|\partial_t^j\partial_x^k u(P_2)-\partial_t^j\partial_x^k u(H_i)\right|$$

$$\times S((\ell+q)\gamma;\tilde{P})\prod_{m=1}^{n} s_2(-k_m\beta_m^{(2)},x_m^{(2)})s_1(-k_m\beta_m^{(1)},\tilde{t}).$$

Here, we denote

$$S(\gamma;P) = s_1(\gamma^{(1)},t)\min_i(s_2(\gamma^{(2)},x_i)),$$

$$S(q\gamma;\tilde{P}) = \min_i\{S(q\gamma;P_1),S(q\gamma;P_2),S(q\gamma;H_i)\},$$

$$s_1(q,\tilde{t}) = \min\{s_1(q,t^{(1)}),s_1(q,t^{(2)})\},$$

$$s_2(q,\tilde{x}_i) = \min_i\{s_2(q,x_i^{(1)}),s_2(q,x_i^{(2)})\}.$$

We take

$$\mu_j^{(\nu)} = \mu_{j_1}^{(\nu)}+\mu_{j_2}^{(\nu)}+\ldots+\mu_{j_n}^{(\nu)},\ \mu_{j_i}^{(\nu)}\ge 0,\ \nu\in\{1,2\},\ \mu_j=(\mu_j^{(1)},\mu_j^{(2)}),\ j\in\{0,1,\ldots,n\}.$$

We study problem (2)–(4) with the following conditions:

1°. For any vector $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$, the following inequality is true:

$$\pi_1|\xi|^2 \le \sum_{i,j=1}^{n} A_{ij}(t,x) s_1(\beta_i^{(1)},t) s_1(\beta_j^{(1)},t) s_2(\beta_i^{(2)},x_i) s_2(\beta_j^{(2)},x_j)\xi_i\xi_j \le \pi_2|\xi|^2,$$

where π_1 and π_2 are fixed positive constants,

$$s_1(\beta_i^{(1)},t) s_1(\beta_j^{(1)},t) s_2(\beta_i^{(2)},x_i) s_2(\beta_j^{(2)},x_j) A_{ij} \in C^{\alpha}(\gamma;\beta;0;Q),$$

$$s_1(\mu_i^{(1)},t) s_2(\mu_i^{(2)},x_i) A_i \in C^{\alpha}(\gamma;\beta;0;Q),$$

$$s_1(\mu_0^{(1)},t)\rho(\mu_0^{(2)},x) A_0 \in C^{\alpha}(\gamma;\beta;0;Q), \qquad A_0 \le K < \infty, \qquad K = \text{const},$$

2°. $f \in C^{\alpha}(\gamma;\beta;\mu_0;Q)$, $\varphi \in C^{2+\alpha}(\tilde{\gamma};\tilde{\beta};0;D)$, $\tilde{\gamma} = (0,\gamma^{(2)})$, $\tilde{\beta} = (0,\beta^{(2)})$, $\partial D \in C^{2+\alpha}$, $\psi \in C^{2+\alpha}(\gamma;\beta;0;Q)$, $\gamma^{(\nu)} = \max\left\{\max_i(1+\beta_i^{(\nu)}), \max_i(\mu_i^{(\nu)} - \beta_i^{(\nu)}), \frac{\mu_0^{(\nu)}}{2}\right\}$, $\nu \in \{1,2\}$, and $\psi(0,x) = \varphi(x)$.

3°. $p_\nu(t,x) \in C^{\alpha}(Q)$, $q_\nu(x) \in C^{2+\alpha}(D)$, $\nu \in \{1,2\}$, functions $F_1(t,x;u(t,x;p;q),p)$, $F_2(x;u(T,x;p;q),q)$ as composite functions of the variables (t,x) and x belong to the spaces $C^{\alpha}(Q)$ and $C^{2+\alpha}(D)$, respectively. Moreover, the functions $f(t,x;p(t,x))$, $F_1(t,x;u(t,x;p;q),p)$, $F_2(x;u(T,x;p;q),q)$, and $\varphi(x,q(x))$ have Hölder derivatives of the second order with respect to arguments u, p, and q, which are continuous as composite functions of the variables (t,x) and x.

The following theorem is true:

Theorem 1. *Suppose that conditions* 1° *and* 2° *are satisfied in the domain* $Q = [0;T) \times D$, *where* D *is a bounded domain from the set* $\mathbb{R}_+^n$. *Then there exists a unique solution of problem (2)–(4) from the space* $C^{2+\alpha}(\gamma;\beta;0;Q)$, *and the estimate*

$$\|u;\gamma;\beta;0;Q\|_{2+\alpha} \le c\left(\|f;\gamma;\beta;\mu_0;Q\|_{\alpha} + \left\|\varphi;\tilde{\gamma};\tilde{\beta};0;D\right\|_{2+\alpha} + \|\psi;\gamma;\beta;0;Q\|_{2+\alpha}\right) \tag{5}$$

is true.

2. Estimation of the Solutions of Boundary-Value Problems with Smooth Coefficients

For the investigation of problem (2)–(4), we first establish the correct solvability of a sequence of auxiliary boundary-value problems with smooth coefficients whose limit value gives the solution of problem (2)–(4).

Let

$$Q_m = Q \cap \{(t,x) \in Q : s_1(1,t) \ge m_1^{-1}, s_2(1,x_i) \ge m_2^{-1}, m = (m_1,m_2), m_1 > 1, m_2 > 1\}, \; i \in \{1,\ldots,n\},$$

be a sequence of domains that converges to $Q^{(0)}$ as $m_1 \to \infty$ and $m_2 \to \infty$.

In the domain Q, we consider the problem of finding a function $u_m(t,x)$ satisfying the equation

$$(L_1 u_m)(t,x) = \left[\partial_t - \sum_{i,j=1}^{n} a_{ij}(t,x)\partial_{x_i}\partial_{x_j} - \sum_{i=1}^{n} a_i(t,x)\partial_{x_i} - a_0(t,x)\right] u_m(t,x)$$

$$= f_m(t,x,p), \tag{6}$$

the initial condition

$$u_m(0,x) = \varphi_m(x,q), \tag{7}$$

and the boundary condition

$$\lim_{x\to z\in\partial D}\left[u_m(t,x) - \psi_m(t,x)\right] = 0. \tag{8}$$

Here, the coefficients a_{ij}, a_i, and a_0 and the functions f_m, φ_m, and ψ_{1m} coincide, for $(t,x)\in Q_m$, with A_{ij}, A_i, and A_0 and f, φ, and ψ, respectively. For $(t,x)\in Q\setminus Q_m$, the coefficients a_{ij}, a_i, and a_0 and the functions f_m, φ_m, and ψ_m are continuous extensions of the coefficients A_{ij}, A_i, and A_0 and the functions f, φ, and ψ from the domain Q_m into domain $Q\setminus Q_m$ [9].

We now estimate the solutions of the auxiliary boundary-value problems. In problem (6)–(8), we pass from the function $u_m(t,x)$ to a new function $v_m(t,x)$ given by the formula

$$u_m(t,x) = v_m(t,x)e^{-\lambda t}, \tag{9}$$

where λ satisfies the inequality $\lambda < -A_0(t,x)$.

Then the function $v_m(t,x)$ is the solution of the boundary-value problem

$$((L_1 - \lambda)v_m)(t,x) = f_m(t,x,p)e^{\lambda t}, \tag{10}$$

$$v_m(0,x) = \varphi_m(x,q), \qquad \lim_{x\to z\in\partial D}[v_m(t,x) - \psi_m(t,x)e^{\lambda t}] = 0. \tag{11}$$

We now formulate the maximum principle for problem (10), (11).

Theorem 2. *Let* $v_m(t,x)$ *be the classical solution of problem (10), (11) in the domain* Q *and let conditions* $1°$ *and* $2°$ *be satisfied. Then the estimate*

$$|v_m| \le \max\left\{\left\|f_m e^{\lambda t}(-\lambda - a_0)^{-1}; Q\right\|_0, \left\|\varphi_m; D\right\|_0, \left\|\psi_m e^{\lambda t}; Q\right\|_0\right\} \tag{12}$$

is true for $v_m(t,x)$.

This theorem is proved by using the same scheme as in the proof of Theorem 2.1 from [4, p. 22], i.e., for the function $v_m(t,x)$, we analyze all possible positions of its positive maximum and negative minimum.

Theorem 3. *Assume that conditions* 1° *and* 2° *are satisfied for problem (10), (11). Then the estimate*

$$\|v_m;\gamma;\beta;0;Q\|_{2+\alpha} \le C\Big(\|f;\gamma;\beta;\mu_0;Q\|_\alpha + \big\|\varphi;\tilde{\gamma};\tilde{\beta};0;D\big\|_{2+\alpha} + \|\psi;\gamma;\beta;0;Q\|_{2+\alpha}\Big) \tag{13}$$

is true for the solution of problem (10), (11).

Proof. In problem (10), (11), we perform the change of variables

$$v_m(t,x) = \omega_m(t,x)e^{-\lambda t} + \psi_m(t,x).$$

Then $\omega_m(t,x)$ is the solution of the problem

$$((L_1-\lambda)\omega_m)(t,x) = f_m(t,x,p)e^{\lambda t} - ((L-\lambda)\psi_m)(t,x) \equiv F_m(t,x),$$

$$\omega_m(0,x) = \varphi_m(x,q) - \psi_m(0,x) \equiv \Phi_m(x), \tag{14}$$

$$\lim_{x\to z\in\partial D} \omega_m(t,x) = 0$$

in the domain Q.

By using the definition of the norm and the interpolation inequalities [8, 11], we find

$$\|\omega_m;\gamma;\beta;0;Q\|_{2+\alpha} \le (1+\varepsilon^\alpha)\langle\omega_m;\gamma;\beta;0;Q\rangle_{2+\alpha} + c(\varepsilon)\|\omega_m;Q\|_0,$$

where ε is an arbitrary real number such that $\varepsilon\in(0,1)$. Therefore, it suffices to estimate the seminorm $\langle\omega_m;\gamma;\beta;0;Q\rangle_{2+\alpha}$. It follows from the definition of seminorm that there exist points P_1, P_2, and H_i of Q for which one of the inequalities

$$\frac{1}{2}\|\omega_m;\gamma;\beta;0;Q\|_{2+\alpha} \le E_p(\omega_m), \qquad p\in\{1,2\},$$

is true. We denote

$$R(\gamma;P) = d_1(\gamma^{(1)},t)\min_i d_2(\gamma^{(2)},x_i),$$

$$d(\beta_i;P) = d_1(\beta_i^{(1)},t)d_2(\beta_i^{(2)},x_i),$$

where

$$d_1(\beta_i^{(1)},t) = \begin{cases} \max\limits_i \left(s_1(\beta_i^{(1)},t), m_1^{-\beta_i^{(1)}} \right), & \beta_i^{(1)} \ge 0, \\ \min\limits_i \left(s_1(\beta_i^{(1)},t), m_1^{-\beta_i^{(1)}} \right), & \beta_i^{(1)} < 0, \end{cases}$$

$$d_2(\beta_i^{(2)},x_i) = \begin{cases} \max\limits_i \left(s_2(\beta_i^{(2)},x_i), m_2^{-\beta_i^{(2)}} \right), & \beta_i^{(2)} \ge 0, \\ \min\limits_i \left(s_2(\beta_i^{(2)},x_i), m_2^{-\beta_i^{(2)}} \right), & \beta_i^{(2)} < 0. \end{cases}$$

If

$$\left| t^{(1)} - t^{(2)} \right| \ge R(2\gamma;\tilde{P})\frac{\tau}{16} \equiv T_1,$$

and τ is an arbitrary number, $\tau \in (0,1)$, then

$$E_2(\omega_m) \le 2\tau^{-\alpha} \| \omega_m;\gamma;\beta;0;Q \|_2. \tag{15}$$

If

$$\left| x_i^{(1)} - x_i^{(2)} \right| \ge n^{-1} R(\gamma;\tilde{P}) d_1(-\beta_i^{(1)}, t^{(2)}) d_2(-\beta_i^{(2)}, \tilde{x}_i) \frac{\tau}{4} \equiv T_2,$$

then

$$E_1(\omega_m) \le 2\tau^{-\alpha} \| \omega_m;\gamma;\beta;0;Q \|_2. \tag{16}$$

Applying the interpolation inequalities to (15) and (16), we find

$$E_p(\omega_m) \le \varepsilon^{\alpha} \| \omega_m;\gamma;\beta;0;Q \|_{2+\alpha} + c(\varepsilon) \| \omega_m;Q \|_0. \tag{17}$$

Suppose that

$$\left| x_i^{(1)} - x_i^{(2)} \right| \le T_2 \quad \text{and} \quad \left| t^{(1)} - t^{(2)} \right| \le T_1.$$

In addition, let $R(\gamma;\tilde{P}) \equiv R(\gamma;P_1)$. We assume that $\left| x^{(1)} - z \right| \ge 2T_2 n$ or $\left| x_n^{(1)} - z_n \right| \ge T_2$, $z \in \partial D$.

We now rewrite problem (14) in the form

$$(L_2\omega_m)(t,x) \equiv \left[\partial_t - \sum_{i,j=1}^{n} a_{ij}(P_1) \partial_{x_i} \partial_{x_j} \right] \omega_m$$

$$= \sum_{i,j=1}^{n} [a_{ij}(P) - a_{ij}(P_1)] \partial_{x_i} \partial_{x_j} \omega_m + \sum_{i=1}^{n} a_i(P) \partial_{x_i} \omega_m$$

$$+ (a_0(P) + \lambda) \omega_m + F_m(t,x) \equiv F_m^{(1)}(t,x,\omega_m) + F_m(t,x),$$

$$\omega_m(0,x) = \Phi_m(x), \qquad \lim_{x \to z \in \partial D} \omega_m(t,x) = 0. \tag{18}$$

Let $V_r^{(1)}$ be a domain from Q and let

$$V_r^{(1)} = \{(t,x) \in Q : |x_i - x_i^{(1)}| \le r T_2,\ i = 1, \ldots, n,\ |t - t^{(1)}| \le r^2 T_1\}.$$

In problem (18), we perform the change of variables

$$\omega_m(t,x) = W_m(t,y), \qquad y_i = d_1(\beta_i^{(1)}, t^{(1)}) d_2(\beta_i^{(2)}, x_i^{(1)}) x_i.$$

As a result, we arrive at the following problem:

$$(L_3 W_m)(t,y) \equiv \Bigg[\partial_t - \sum_{i,j=1}^{n} d_1(\beta_i^{(1)}, t^{(1)}) d_1(\beta_j^{(1)}, t^{(1)}) d_2(\beta_i^{(2)}, x_i^{(1)}) d_2(\beta_j^{(2)}, x_j^{(1)})$$

$$\times a_{ij}(P_1) \partial_{y_i} \partial_{y_j} \Bigg] W_m = F_m^{(1)}(t, \tilde{y}, W_m) + F_m(t, \tilde{y}),$$

$$W_m(0,y) = \Phi_m(\tilde{y}), \qquad W_m\big|_{\Gamma} = 0,$$

where

$$\tilde{y} = (d_1(-\beta_1^{(1)}, t^{(1)}) d_2(-\beta_1^{(2)}, x_1^{(1)}) y_1, \ldots, d_1(-\beta_n^{(1)}, t^{(1)}) d_2(-\beta_n^{(2)}, x_n^{(1)}) y_n).$$

We denote

$$y_i^{(1)} = d_1(\beta_i^{(1)}, t^{(1)}) d_2(\beta_i^{(2)}, x_i^{(1)}) x_i^{(1)},$$

$$H_r^{(1)} = \{(t,y) : |y_i - y_i^{(1)}| \le r n^{-1} \sqrt{T_1},\ i \in \{1, \ldots, n\}, |t - t^{(1)}| \le r T_1\}$$

and choose a three times differentiable function $\eta(t,y)$ with the following properties:

$$\eta(t,y) = \begin{cases} 1, & (t,y) \in H_{1/4}^{(1)}, \quad 0 \le \eta(t,y) \le 1, \\ 0, & (t,y) \notin H_{3/4}^{(1)}, \quad \left|\partial_t^j \partial_y^k \eta\right| \le C_{kj} R(-(2j + |k|)\gamma; P_1). \end{cases}$$

Then the function $V_m(t,y)=\eta(t,y)W_m(t,y)$ is a solution of the boundary-value problem

$$
\begin{aligned}
(L_3V_m)(t,y) = {} & \sum_{i,j=1}^{n} d_1(\beta_i^{(1)},t^{(1)})d_1(\beta_j^{(1)},t^{(1)})d_2(\beta_i^{(2)},x_i^{(1)})d_2(\beta_j^{(2)},x_j^{(1)}) \\
& \times a_{ij}(P_1)[\partial_{y_i}\eta\partial_{y_j}W_m+\partial_{y_j}\eta\partial_{y_i}W_m] \\
& + w_m\left[\sum_{i,j=1}^{n} d_1(\beta_i^{(1)},t^{(1)})d_1(\beta_j^{(1)},t^{(1)})d_2(\beta_i^{(2)},x_i^{(1)})\right. \\
& \left.\times d_2(\beta_j^{(2)},x_j^{(1)})a_{ij}(P_1)\partial_{y_i}\partial_{y_j}\eta-\partial_t\eta\right] \\
& +\eta(F_m^{(1)}+F_m) \equiv F_m^{(2)}(t,\tilde{y},\eta,W_m)+\eta F_m(t,\tilde{y}), \\
& V_m(0,y) = \eta\Phi_m(\tilde{y}), \qquad V_m\big|_{\Gamma} = 0.
\end{aligned}
\tag{19}
$$

By using Theorem 5.2 [4, p. 364], for the solution of problem (19), we obtain

$$
\begin{aligned}
E_3(V_m) \equiv {} & d^{-\alpha}(N_1,N_2)\left|\partial_t^j\partial_y^k V_m(N_1)-\partial_t^j\partial_y^k V_m(N_2)\right| \\
& \leq C\left(\left\|F_m^{(2)}+\eta F_m\right\|_{C^{\alpha}(H_{3/4}^{(1)})}+\left\|\eta\Phi_m\right\|_{C^{2+\alpha}(H_{3/4}^{(1)})}\right),
\end{aligned}
\tag{20}
$$

where $(N_1,N_2)\subset H_{1/4}^{(1)}$ and $d(N_1,N_2)$ is the parabolic distance between the points N_1, N_2, $2j+|k|=2$.

In view of the properties of the function $\eta(t,y)$, we find

$$
\begin{aligned}
\left\|F_m^{(2)}+\eta F_m\right\|_{C^{\alpha}(H_{3/4}^{(1)})} \leq {} & cR(-(2+\alpha)\gamma;P_1)\Big(\left\|W_m;\gamma;0;0;H_{3/4}^{(1)}\right\|_2 \\
& +\left\|W_m;H_{3/4}^{(1)}\right\|_0+\left\|F_m^{(1)};\gamma;0;2\gamma;H_{3/4}^{(1)}\right\|_{\alpha}+\left\|F_m;\gamma;0;2\gamma;H_{3/4}^{(1)}\right\|_{\alpha}\Big), \\
\left\|\eta\Phi_m\right\|_{C^{2+\alpha}(H_{3/4}^{(1)})} \leq {} & R(-(2+\alpha)\gamma;P_1)\Big(\lambda_0\left\|W_m;\gamma;0;0;H_{3/4}^{(1)}\right\|_{2+\alpha} \\
& +c\left\|W_m;H_{3/4}^{(1)}\right\|_0+c_1\left\|\Phi_m;\gamma;0;0;H_{3/4}^{(1)}\right\|_{2+\alpha}\Big).
\end{aligned}
\tag{21}
$$

Substituting (21) in (20) and returning to the variables (t,x), we obtain

$$E_p(\omega_m) \le c_1\Big(\big\|F_m^{(1)};\gamma;\beta;2\gamma;V_{3/4}^{(1)}\big\|_\alpha + \big\|\Phi_m;\gamma;\beta;0;V_{3/4}^{(1)}\big\|_{2+\alpha}$$

$$+\big\|F_m;\gamma;\beta;2\gamma;V_{3/4}^{(1)}\big\|_\alpha + \big\|\omega_m;\gamma;\beta;0;V_{3/4}^{(1)}\big\|_2 + \big\|\omega_m;V_{3/4}^{(1)}\big\|_0\Big)$$

$$+\lambda_0\big\|\omega_m;\gamma;\beta;0;V_{3/4}^{(1)}\big\|_{2+\alpha}.$$

By using the interpolation inequalities and estimates of the norm for each term in the expressions for $F_m^{(1)}$, F_m, and Φ_m, we obtain

$$E_p(\omega_m) \le (\lambda_0 + \varepsilon^\alpha(n+2) + r^2 n^2)\big\|\omega_m;\gamma;\beta;0;H_{3/4}^{(1)}\big\|_{2+\alpha}$$

$$+c_2\big\|\omega_m;H_{3/4}^{(1)}\big\|_0 + c_3\Big(\|f_m;\gamma;\beta;\mu_0;Q\|_\alpha$$

$$+\big\|\varphi_m;\tilde\gamma;\tilde\beta;0;D\big\|_{2+\alpha} + \|\psi_{1m};\gamma;\beta;0;Q\|_{2+\alpha}\Big).$$

Consider the case where $|x^{(1)} - z| \le 2T_2 n$ and $|x_n^{(1)} - z_n| \le T_2$, $z \in \partial D$. Let $K(P)$ be a ball of radius R_0, $R_0 \ge 4(T_2 n + T_1)$ centered at a point $P \in \Gamma$ and containing the points P_1, P_2, and H_i. By using the restrictions imposed on the smoothness of the boundary ∂D, we can straighten the domain $\partial D \cap K(P)$ with the help of a one-to-one transformation $x = \psi_3(\xi)$ [11, p. 126]. As a result of this transformation, the domain $D \cap K(P)$ turns into a domain Π whose points are such that $\xi_n \ge 0$. Assume that, under this transformation $\omega_m(t,x)$, P_1, P_2, and H_i pass into $\omega_m^{(1)}(t,\xi)$, M_1, M_2, and $N_i^{(1)}$, respectively.

We denote the coefficients of the differential expression $(L_1 - \lambda)$ in the domain Π by $p_{ij}(\tau,\xi)$, $p_i(\tau,\xi)$, and $p_0(\tau,\xi)$ and the coefficients of the nonlocal condition by $q_k^{(1)}(\xi)$. Then $\omega_m^{(1)}(t,\xi)$ is a solution of the problem

$$(L_4\omega_m^{(1)})(t,\xi) \equiv \left[\partial_t - \sum_{i,j=1}^n p_{ij}(M_1)\partial_{\xi_i}\partial_{\xi_j}\right]\omega_m^{(1)}$$

$$= \sum_{i,j=1}^n [p_{ij}(t,\xi) - p_{ij}(M_1)]\partial_{\xi_i}\partial_{\xi_j}\omega_m^{(1)} + \sum_{i=1}^n p_i(t,\xi)\partial_{\xi_i}\omega_m^{(1)}$$

$$+(p_0(t,\xi)+\lambda)\omega_m^{(1)} + F_m(t,\psi_1(\xi))$$

$$\equiv F_m^{(2)}(t,\xi,\omega_m^{(1)}) + F_m(t,\psi_3(\xi)),$$

$$\omega_m^{(1)}(0,\xi) = \Phi_m(\psi_1(\xi)) = \Phi_m^{(1)}(\xi,\omega_m^{(1)}),$$

$$\omega_m^{(1)}\Big|_{\xi_n=0} = 0.$$

By virtue of Theorem 6.1 [4, p. 368], we arrive at the inequality

$$E_\ell(\omega_m^{(1)}) \le (\lambda_0 + \varepsilon^\alpha (n+2) + r^2 n^2)\left\|\omega_m^{(1)};\gamma;\beta;0;\Pi\right\|_{2+\alpha}$$

$$+ c_4 \left\|\omega_m^{(1)};\Pi\right\|_0 + c_5 \Big(\|f_m;\gamma;\beta;\mu_0;Q\|_\alpha$$

$$+\left\|\varphi_m;\tilde{\gamma};\tilde{\beta};0;D\right\|_{2+\alpha} + \|\psi_{1m};\gamma;\beta;0;Q\|_{2+\alpha}\Big).$$

Choosing sufficiently small ε and r, we find

$$\|v_m;\gamma;\beta;0;Q\|_{2+\alpha} \le c\Big(\|f_m;\gamma;\beta;\mu_0;Q\|_\alpha$$

$$+\left\|\varphi_m;\tilde{\gamma};\tilde{\beta};0;D\right\|_{2+\alpha} + \|\psi_{1m};\gamma;\beta;0;Q\|_{2+\alpha}\Big). \tag{22}$$

In view of inequalities (22) and the estimates

$$\|f_m;\gamma;\beta;\mu_0;Q\|_\alpha \le c\|f;\gamma;\beta;\mu_0;Q\|_\alpha,$$

$$\left\|\varphi_m;\tilde{\gamma};\tilde{\beta};0;D\right\|_{2+\alpha} \le c\left\|\varphi;\tilde{\gamma};\tilde{\beta};0;D\right\|_{2+\alpha},$$

$$\|\psi_{1m};\gamma;\beta;0;Q\|_{2+\alpha} \le c\|\psi_1;\gamma;\beta;0;Q\|_{2+\alpha},$$

we arrive at inequality (13).

Proof of Theorem 1. By using the change of variables $v_m = u_m e^{\lambda t}$ and inequality (13), we arrive at the following estimate for the solution of problem (6)–(8):

$$\|u_m;\gamma;\beta;0;Q\|_{2+\alpha} \le c\Big(\|f;\gamma;\beta;\mu_0;Q\|_\alpha + \left\|\varphi;\tilde{\gamma};\tilde{\beta};0;D\right\|_{2+\alpha} + \|\psi;\gamma;\beta;0;Q\|_{2+\alpha}\Big), \tag{23}$$

whose right-hand side is independent of m_1 and m_2. In addition, the sequences

$$\{U_m^{(0)}\} \equiv \{u_m(P)\},$$

$$\{U_m^{(1)}\} \equiv \{R(\gamma;P) d_1(-\beta_i^{(1)},t) d_2(-\beta_i^{(2)},x_i)\partial_{x_i} u_m(P)\},$$

$$\{U_m^{(2)}\} \equiv \{R(2\gamma;P)\partial_t u_m(P)\},$$

$$\{U_m^{(3)}\} \equiv \{R(2\gamma;P) d_1(-\beta_i^{(1)},t) d_1(-\beta_j^{(1)},t) d_2(-\beta_i^{(2)},x_i) d_2(-\beta_j^{(2)},x_j)\partial_{x_i}\partial_{x_j} u_m(P)\}$$

are uniformly bounded and equicontinuous in the domain $\bar{Q}$. According to the Arzela theorem, there exist subsequences $\{U_m^{(\nu)}\}$ uniformly convergent in $\bar{Q}$ to $U^{(\nu)}$, $\nu \in \{0,1,2,3\}$. Passing to the limit as $m_1 \to \infty$ and $m_2 \to \infty$ in problem (6)–(8), we conclude that $u(t,x) = U^{(0)}$ is the unique solution of problem (2)–(4), $u \in C^{\alpha}(\gamma;\beta;0;Q)$.

3. Optimal Control Problem

In a domain $Q = [0,T) \times D$, we consider the problem of finding functions $u(t,x)$, $p(t,x)$, and $q(x)$ for which functional (1) reaches its minimum in the class of bounded functions $(p,q) \in V$, while the function $u(t,x;p(t,x),q(x))$ with $(t,x) \in Q^{(0)} = Q \backslash Q_{(0)}$, $Q_{(0)} = \{(t,x) \in Q : t = t_0, x \in D \backslash \Omega\} \bigcup \{(t,x) \in Q : t \in [0,T), x \in \Omega\}$ is a solution of the boundary-value problem (2)–(4).

We investigate problem (1)–(4) in the case where conditions 1°–3° are satisfied. In order to establish the existence of solution of problem (1)–(4), it is necessary first to prove the solvability of the auxiliary problems with smooth coefficients.

In the domain Q, we consider the problem of finding functions (u_m, p, q) on which the functional

$$I(p,q) = \int_0^T dt \int_D F_1(t,x;u_m(t,x;p;q),p)\,dx$$

$$+ \int_D F_2(x;u_m(T,x;p;q),q)\,dx \tag{24}$$

attains its minimum in the class of bounded functions $(p,q) \in V$ in which $u_m(t,x)$ is the solution of boundary-value problem

$$(L_1 u_m)(t,x) = f_m(t,x;p), \tag{25}$$

$$u_m(0,x) = \varphi_m(x;q), \tag{26}$$

$$(u_m - \psi_m)(t,z) \equiv 0\,. \tag{27}$$

By using the Green function $(G_m^{(1)}, \Gamma_m)$ from [5, p. 62], for problem (25)–(27), we conclude that, for any $(p,q) \in V$, there exists a solution of this problem, which is unique and given by the formula

$$u_m(t,x,p,q) = \int_0^T d\tau \int_D G_m^{(1)}(t,x,\tau,\xi) f_m(\tau,\xi,p)\,d\xi$$

$$+ \int_D G_m^{(1)}(t,x,0,\xi)\varphi_m(\xi,q)\,d\xi$$

$$+\int_0^T d\tau \int_{\partial D} \Gamma(t,x,\tau,\xi)\psi_m(\tau,\xi)d\xi. \tag{28}$$

We denote

$$\mu(\tau,\xi) = \int_\tau^T dt \int_D G_m^{(1)}(t,x,\tau,\xi)\frac{\partial F_1(t,x,u_m,p)}{\partial u_m}dx$$

$$+\int_D G_m^{(1)}(T,x,\tau,\xi)\frac{\partial F_2(x,u_m,q)}{\partial u_m}dx,$$

$$\eta(\xi) = \int_0^T dt \int_D G_m^{(1)}(t,x,0,\xi)\frac{\partial F_1(t,x,u_m,p)}{\partial u_m}dx$$

$$+\int_D G_m^{(1)}(T,x,0,\xi)\frac{\partial F_2(x,u_m,q)}{\partial u_m}dx,$$

$$H_\mu(\mu,u_m,p) = F_1(t,x;u_m,p)+\mu(t,x)f_m(t,x;p),$$

$$H_\eta(\eta,u_m,q) = F_2(x;u_m,q)+\eta(x)\varphi_m(x;q).$$

The following theorems are true:

Theorem 4. *Assume that conditions* 1°–3° *are satisfied. Then*

(i) if H_μ *and* H_η *are monotonically increasing functions of the arguments* p *and* q, *respectively, then* $p_1(t,x)$ *and* $q_1(x)$ *are optimal controls and the optimal solution of problem (24)–(27) has the form*

$$u_m^{(0)}(t,x;p,q) = u_m(t,x;p_1,q_1);$$

(ii) if H_μ *is a monotonically increasing function of the argument* p *and* H_η *is a monotonically decreasing function of the argument* q, *then the functions* $p_1(t,x)$ *and* $q_2(x)$ *are optimal controls and the optimal solution of problem (24)–(27) has the form*

$$u_m^{(0)}(t,x;p,q) = u_m(t,x;p_1,q_2);$$

(iii) if H_μ *is a monotonically decreasing function of the argument* p *and* H_η *is a monotonically increasing function of the argument* q, *then* $p_2(t,x)$ *and* $q_1(x)$ *are optimal controls and*

$$u_m^{(0)}(t,x;p,q) = u_m(t,x;p_2,q_1)$$

is the optimal solution of problem (24)–(27);

(iv) if H_μ *and* H_η *are monotonically decreasing functions of the arguments* p *and* q, *respectively, then* $p_2(t,x)$ *and* $q_2(x)$ *are optimal controls and the optimal solution of problem (24)–(27) has the form*

$$u_m^{(0)}(t,x;p,q) = u_m(t,x;p_2,q_2).$$

We now establish the conditions of existence of the optimal solution of problem (24)–(27) in the case where the functions H_λ and H_μ are not monotonic.

Theorem 5. *Suppose that conditions* 1°–3° *are satisfied for problem (24)–(27) and the functions* H_μ *and* H_η *are not monotonic with respect to the arguments* p *and* q, *respectively.*

Then, in order that the control $(p^{(0)},q^{(0)})\in V$ *and the corresponding solution* $u_m(t,x,p^{(0)},q^{(0)})$ *of the boundary-value problem (24)–(27) be optimal, it is necessary and sufficient that the following conditions be satisfied:*

(i) as a function of the argument p, *the function* $H_\mu(t,x,p)$ *takes its minimal value at the point* $p^{(0)}$;

(ii) as a function of the argument q, *the function* $H_\eta(x,q)$ *takes its minimal value at the point* $q^{(0)}$;

(iii) for any nonzero vector $\bar{\xi}=(\xi_1,\xi_2)$ *and* $(t,x)\in\bar{Q}$, *the following inequality is satisfied:*

$$K_1(t,x,\bar{\xi}) \equiv \partial_{u_m}^2 F_1(t,x;u_m^{(0)},p^{(0)})\xi_1^2 + 2\partial_{u_m}\partial_p F_1(t,x;u_m^{(0)},p^{(0)})\xi_1\xi_2$$

$$+\partial_p^2 F_1(t,x;u_m^{(0)},p^{(0)})\xi_2^2 > 0;$$

(iv) for any vector $\bar{y}=(y_1,y_2)$ *and* $x\in D$, *the following inequality is satisfied:*

$$K_2(x,\bar{y}) \equiv \partial_{u_m}^2 F_2(x;u_m^{(0)},q^{(0)})y_1^2 + 2\partial_{u_m}\partial_q F_2(x;u_m^{(0)},q^{(0)})y_1y_2$$

$$+\partial_q^2 F_2(x;u_m^{(0)},q^{(0)})y_2^2 > 0.$$

Proof. Theorems 4 and 5 are proved by using the procedure presented in [8]. Passing to the limit in problem (24)–(27) as $m_1\to\infty$ and $m_2\to\infty$, we get the optimal solution of problem (1)–(4).

Note that it is also possible to establish the corresponding theorems for the case where one of the functions is monotonic and the other is not monotonic.

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