Characterization of compact subspaces of the space of separately continuous functions with the cross-uniform topology

Ivasiuk Roman, Maslyuchenko Oleksandr

ivasiuk.roman@chnu.edu.ua, ovmasl@gmail.com
Yuriy Fedkovych Chernivtsi National University
University of Silesia in Katowice

Let X and Y be a topological space and Z be a metric space. The space $S(X \times Y, Z)$ of separately continuous function $f : X \times Y \to Z$ we always endows by *the cross-uniform topology*. This topology is generated by the base which consists of the sets

$$W_{E,\varepsilon}(f_0) = \Big\{ f \in S(X \times Y, Z) : |f(p) - f_0(p)|_Z < \varepsilon \text{ for any } p \in \mathrm{cr}E \Big\},$$

where E is a finite subset of $X \times Y$,

$$\mathrm{cr}E = (X \times \mathrm{pr}_Y(E)) \cup (\mathrm{pr}_X(E) \times Y)$$

is *the cross* of the set E, $\varepsilon > 0$ and $f_0 \in S(X \times Y)$. This topology was introduced in [1] for the space $S = S([0;1]^2, \mathbb{R})$ and called there by *the topology of the sectionally uniform convergence.* In [1] it was proved only that S is a separable non-metrizable complete topological vector space, and the authors asked about the other properties of S. In [2, 3, 4] the authors proved that $S(X \times Y)$ is a meager, compleat, barreled and bornological topological vector space for any compacta X and Y without isolated points.

Let $w(X)$ denote the weight of a topological space X and let $c(X)$ denote the cellularity of X. *The sharp cellularity* is

$$c^\sharp(X) = \sup \Big\{ |\mathcal{U}|^+ : \mathcal{U} \text{ is a disjoint family of open sets in } X \Big\},$$

where $|A|$ means the cardinality of a set A and $\mathfrak{m}^+$ means the least cardinal number which is grater then the cardinal number $\mathfrak{m}$. In [5] was announced the following result.

Theorem 1. *Let X, Y be infinity compacta and K be a compact. Then K embeds into $S(X \times Y)$ if and only if $w(K) < \min\{c^\sharp(X), c^\sharp(Y)\}$.*

Now we pass to the generalization of this result to the space $S(X \times Y, Z)$. Recall that a compact space K calls an Eberlein compact if it is homeomorphic to some subspace of $C_p(X)$ for some compact space X. We start from the following observation.

Theorem 2. *Let X be a compact space, Y be a separable metrizable space and K be a compact subspace of $C_p(X, Y)$. Then K is an Eberlein compact.*

This result allows to modify the proof of Theorem 1 and obtain the following.

Theorem 3. *Let X, Y be compacta, Z be a separable metric space and K be a compact space which embeds into $S(X \times Y, Z)$ Therefore, the following inequality holds: $w(K) < \min\{c^\sharp(X), c^\sharp(Y)\}$.*

And now we obtain the following generalization of Theorem 1.

Theorem 4. *Let X, Y be infinity compacta, Z be a separable metric space which contains a homeomorphic copy of $\mathbb{R}$ and K be a compact. Then K embeds into $S(X \times Y, Z)$ if and only if $w(K) < \min\{c^\sharp(X), c^\sharp(Y)\}$.*

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