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Equi-Baire 1 families of functions

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ABSTRACT

We study equi-Baire 1 families of functions between Polish spaces X and Y. We show that the respective ε-gauge in the definition of such a family can be chosen upper semi-continuous. We prove that a pointwise convergent sequence of continuous functions forms an equi-Baire 1 family. We study families of separately equi-Baire 1 functions of two variables and show that the family of all sections of separately continuous functions also forms an equi-Baire 1 family. We characterize equi-Baire 1 families of characteristic functions. This leads us to an example witnessing that the exact counterpart of the Arzelà-Ascoli theorem for families of real-valued Baire 1 functions on $[0, 1]$ is false. Also, the weak counterpart of this theorem dealing with restrictions to nonmeager sets is false. On the other hand, we obtain a simple proof of the Arzelà-Ascoli type theorem for sequences of equi-Baire 1 real-valued functions in the case of pointwise convergence.

1. Introduction

A function f between metric spaces X and Y is called *Baire 1* if it F_σ-measurable, that is, whenever the preimage of any open set is F_σ (cf. [11, Definition 24.1]). Note that, in the traditional terminology, the notion of Baire 1 function was reserved to the limit of pointwise convergent sequence of continuous functions and F_σ-measurable functions were called of Borel class 1. In special cases, for instance if $Y = \mathbb{R}$, these two families coincide. In general, every limit of pointwise convergent sequence of continuous functions is F_σ-measurable but the converse need not hold (cf. [22, pp. 90–91], [11, Theorem 24.10]).

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We denote $\mathbb{R}_+ := (0, \infty)$. The following epsilon-delta characterization of Baire 1 functions was obtained in [15] (see also [14], [2], [20], [7]).

Theorem 1.1. *Let (X, d_X) be a separable metric space, and (Y, d_Y) be a Polish space. A function $f\colon X \to Y$ is Baire 1 if and only if, for each $\varepsilon > 0$, there exists a function $\delta_\varepsilon^f\colon X \to \mathbb{R}_+$ (called an ε-gauge) such that, for all $x, x' \in X$, the condition $d_X(x, x') < \min\{\delta_\varepsilon^f(x), \delta_\varepsilon^f(x')\}$ implies $d_Y(f(x), f(x')) \le \varepsilon$.*

We are interested in the Baire 1 analogue of equi-continuity. This notion was invented by Lecomte [14]. Recently, it has been rediscovered by Alighani-Koopaei [1]. The idea comes from the characterization stated in Theorem 1.1.

Definition 1.2. Let (X, d_X), (Y, d_Y) be metric spaces. We say that a family $\mathcal{F}$ of functions from X to Y is *equi-Baire 1* if, for each $\varepsilon > 0$, there exists a (Baire 1) function $\delta_\varepsilon\colon X \to \mathbb{R}_+$ such that, for all $f \in \mathcal{F}$ and $x, x' \in X$, the condition $d_X(x, x') < \min\{\delta_\varepsilon(x), \delta_\varepsilon(x')\}$ implies $d_Y(f(x), f(x')) \le \varepsilon$.

By Theorem 1.1, if X is a separable metric space and Y is a Polish space, then every member of an equi-Baire 1 family of functions from X to Y is a Baire 1 function.

The following characterization of equi-Baire 1 families is due to Lecomte [14, Proposition 32]. For simplicity, we have slightly changed its formulation.

Theorem 1.3. *Let (X, d_X) be a separable metric space, and (Y, d_Y) be a Polish space. For a family $\mathcal{F}$ of functions from X to Y, the following conditions are equivalent:*

(i) *$\mathcal{F}$ is equi-Baire 1;*
(ii) *for every $\varepsilon > 0$, there is a cover $(X_i)_{i\in\mathbb{N}}$ of X consisting of closed sets such that $\operatorname{diam} f[X_i] \le \varepsilon$ for all $i \in \mathbb{N}$ and $f \in \mathcal{F}$;*
(iii) *there is a finer metrizable separable topology on X making $\mathcal{F}$ equi-continuous;*
(iv) *every nonempty closed subset E of X contains a point x such that the family $\{f|_E\colon f \in \mathcal{F}\}$ is equi-continuous at x.*

Equivalence (i)⇔(ii) was rediscovered later by Alikhani-Koopaei [1, Theorem 3.6]. Condition (ii) formulated for a one-element family has been known as a characterization of a Baire 1 function in various cases. It appeared in [8] for $X = Y = \mathbb{R}$, and in [5, Proposition 1.1] if X is a topological space and Y is a separable metric space. A more general result, where Y can be non-separable, was shown in [10, Lemma 2].

A definition equivalent to (iv) was formulated by Glasner and Megrelishvili in [9, Def. 6.8] under the name *barely continuous family*. They used it to identify certain subclasses of discrete dynamical systems.

2. How good can ε-gauge for equi-Baire 1 families be?

It was observed in [14, Corollary 33] that a gauge δ_ε^f, used in Theorem 1.1, can be chosen Baire 1. Moreover, if X is compact and $f\colon X \to \mathbb{R}$ is a bounded Baire 1, then (see [2]) δ_ε^f can be chosen upper semi-continuous and its oscillation index $\beta(\delta)$ is finite. A related result for unbounded functions and $X = [0, 1]$ was obtained in [16, Thm. 4]. It can be observed that the argument from [16] works for any Polish space. We give a slightly modified proof for the reader's convenience.

Let us recall the notion of oscillation rank (cf. [12], [17], [6]). Denote by CL the family of all closed subsets of a fixed Polish space X. Given $f\colon X \to \mathbb{R}$, $F \in \mathrm{CL}$ and $x \in F$, denote by $\operatorname{osc}(f, x, F)$ the oscillation of $f|_F$ at a point x (defined as $\inf_V(\operatorname{diam} f[V \cap F])$ where infimum is taken over all open neighbourhoods V of x). For a fixed $\varepsilon > 0$, denote

$$D(f,\varepsilon,F) := \{x \in F\colon\ \mathrm{osc}(f,x,F) \geq \varepsilon\} \in \mathrm{CL}\,.$$

Then $D(\cdot) := D(f,\varepsilon,\cdot)$ is a derivation on CL (see [6]). Put $F^0_{f,\varepsilon} := X$, $F^{\alpha+1}_{f,\varepsilon} := D(F^\alpha_{f,\varepsilon})$ for $\alpha < \omega_1$ and $F^\alpha_{f,\varepsilon} := \bigcap_{\xi<\alpha} F^\xi_{f,\varepsilon}$ if $\alpha < \omega_1$ is a limit ordinal. The rank $\beta(f,\varepsilon)$ is defined as the smallest ordinal α such that $F^\alpha_{f,\varepsilon}(X) = \emptyset$ if such an α exists, and ω_1, otherwise. The oscillation rank of f is defined by $\beta(f) := \sup_{\varepsilon>0} \beta(f,\varepsilon)$. It is known that, for a Baire 1 function f, we have $\beta(f) < \omega_1$ (see [12] and [6]).

Theorem 2.1 *(essentially Lindner & Lindner). Given a Polish space (X,d) and a function $f\colon X \to \mathbb{R}$ with $\beta(f) = \alpha < \omega_1$, for every $\varepsilon > 0$ there exists an upper semi-continuous gauge $\delta_\varepsilon\colon X \to \mathbb{R}_+$ such that $\beta(\delta_\varepsilon) \leq \alpha$ and for all $x, x' \in X$, the condition $d(x,x') < \min\{\delta_\varepsilon(x), \delta_\varepsilon(x')\}$ implies $|f(x) - f(x')| \leq \varepsilon$.*

Proof. Let $\varepsilon > 0$. Consider a decreasing transfinite sequence $(F^\xi_{f,\varepsilon})_{\xi<\alpha}$ of closed sets defined as in the definition of $\beta(f,\varepsilon)$. Note that $X = \bigcup_{\xi<\alpha}(F^\xi_{f,\varepsilon} \setminus F^{\xi+1}_{f,\varepsilon})$.

We will define a function $\delta_\varepsilon\colon X \to \mathbb{R}_+$ on each set $F^\xi_{f,\varepsilon} \setminus F^{\xi+1}_{f,\varepsilon}$. Fix $\xi < \alpha$ and $t \in F^\xi_{f,\varepsilon} \setminus F^{\xi+1}_{f,\varepsilon}$. Then $\mathrm{osc}(f,t,F^\xi_{f,\varepsilon}) < \varepsilon$. So, pick an open neighbourhood V_t of t with $\mathrm{diam}\, f[V_t \cap F^\xi_{f,\varepsilon}] < \varepsilon$. Define

$$\delta^t(x) := \frac{1}{2} \min\{1, d(x, X \setminus V_t)\} \text{ whenever } x \in F^\xi_{f,\varepsilon} \setminus F^{\xi+1}_{f,\varepsilon}.$$

Then δ^t is a (1/2)-Lipschitz function on $F^\xi_{f,\varepsilon} \setminus F^{\xi+1}_{f,\varepsilon}$. Now, let

$$\delta_\varepsilon(x) := \sup\{\delta^t(x)\colon t \in F^\xi_{f,\varepsilon} \setminus F^{\xi+1}_{f,\varepsilon}\} \text{ whenever } x \in F^\xi_{f,\varepsilon} \setminus F^{\xi+1}_{f,\varepsilon}.$$

This defines a positive function on X.

Note that δ_ε restricted to any set $F^\xi_{f,\varepsilon} \setminus F^{\xi+1}_{f,\varepsilon}$ is continuous as an upper bound of equi-continuous functions on $F^\xi_{f,\varepsilon} \setminus F^{\xi+1}_{f,\varepsilon}$. Also, δ_ε restricted to $F^\xi_{f,\varepsilon}$ is continuous at every point of $F^\xi_{f,\varepsilon} \setminus F^{\xi+1}_{f,\varepsilon}$ since this last set is open in $F^\xi_{f,\varepsilon}$.

We claim that $\beta(\delta_\varepsilon) \leq \alpha$. To prove it, fix $\eta > 0$ and consider the respective decreasing transfinite sequence $(F^\xi_{\delta_\varepsilon,\eta})_{\xi<\omega_1}$ of closed sets corresponding to the definition of $\beta(\delta_\varepsilon,\eta)$. By transfinite induction we show that $F^\xi_{\delta_\varepsilon,\eta} \subseteq F^\xi_{f,\varepsilon}$ for all $\xi < \omega_1$. For $\xi = 0$ it is obvious. Suppose the inclusion holds for an ordinal $\xi < \omega_1$. Since $\delta_\varepsilon|_{F^\xi_{f,\varepsilon}}$ is continuous on $F^\xi_{f,\varepsilon} \setminus F^{\xi+1}_{f,\varepsilon}$, we have $F^{\xi+1}_{\delta_\varepsilon,\eta} \subseteq F^{\xi+1}_{f,\varepsilon}$. In the case when ξ is a limit ordinal, the inclusion also holds. This yields

$$\beta(\delta_\varepsilon,\eta) \leq \beta(f,\varepsilon) \leq \beta(f) < \alpha.$$

Hence $\beta(\delta_\varepsilon) \leq \alpha$, since $\eta > 0$ has been taken arbitrarily.

We claim that for any $\xi < \omega_1$ and $x \in F^\xi_{f,\varepsilon} \setminus F^{\xi+1}_{f,\varepsilon}$,

$$\text{the ball } B(x,\delta_\varepsilon(x)) \text{ is disjoint from } F^{\xi+1}_{f,\varepsilon}. \tag{1}$$

Indeed, fix $\xi < \omega_1$ and $x \in F^\xi_{f,\varepsilon} \setminus F^{\xi+1}_{f,\varepsilon}$. By the definition of δ_ε pick $t \in F^\xi_{f,\varepsilon} \setminus F^{\xi+1}_{f,\varepsilon}$ with $\delta^t(x) > \frac{3}{4}\delta_\varepsilon(x)$. Recall that $\mathrm{diam}\, f[V_t \cap F^\xi_{f,\varepsilon}] < \varepsilon$ and $d(x, X \setminus V_t) \geq 2\delta^t(x) > \frac{3}{2}\delta_\varepsilon(x)$. Therefore $B(x,\delta_\varepsilon(x)) \subseteq V_t$ and $V_t \cap F^{\xi+1}_{f,\varepsilon} = \emptyset$ which gives (1).

We will show that δ_ε is an ε-gauge for f. Let $x, x' \in X$ and $d(x,x') < \min\{\delta(x), \delta(x')\}$. Assume that $x \in F^\gamma_{f,\varepsilon} \setminus F^{\gamma+1}_{f,\varepsilon}$ and $x' \in F^\xi_{f,\varepsilon} \setminus F^{\xi+1}_{f,\varepsilon}$ where $\gamma \leq \xi < \alpha$. We will consider two cases:

(i) $\gamma < \xi$;
(ii) $\gamma = \xi$.

In the first case, by (1) we have $B(x', \delta_\varepsilon(x')) \cap F^{\gamma+1}_{f,\varepsilon} = \emptyset$. Hence $x' \notin F^{\gamma+1}_{f,\varepsilon}$ which gives a contradiction with $x' \in F^{\xi}_{f,\varepsilon} \subseteq F^{\gamma+1}_{f,\varepsilon}$.

Thus $\gamma = \xi$ (the second case). Pick $t \in F^{\xi}_{f,\varepsilon} \setminus F^{\xi+1}_{f,\varepsilon}$ such that $\delta^t(x) > \frac{3}{4}\delta_\varepsilon(x)$. So,

$$d(x, X \setminus V_t) \geq 2\delta^t(x) > \frac{3}{2}\delta_\varepsilon(x) > d(x, x').$$

Hence $x, x' \in V_t$. By the choice of V_t we have $|f(x) - f(x')| < \varepsilon$, as desired.

Finally, we prove that δ_ε is upper semi-continuous. Let $x \in X$ and take any sequence (x_n) in X convergent to x with $\lim_n \delta_\varepsilon(x_n) = g$. We want to prove that $g \leq \delta_\varepsilon(x)$. Pick $\xi < \alpha$ such that $x \in F^{\xi}_{f,\varepsilon} \setminus F^{\xi+1}_{f,\varepsilon}$. Since $F^{\xi+1}_{f,\varepsilon}$ is a closed set and $x \notin F^{\xi+1}_{f,\varepsilon}$, we can suppose that $x_n \notin F^{\xi+1}_{f,\varepsilon}$ for all n. Firstly, assume that all but finitely many x_n's belong to $F^{\xi}_{f,\varepsilon} \setminus F^{\xi+1}_{f,\varepsilon}$. Then $g = \lim_n \delta_\varepsilon(x_n) = \delta_\varepsilon(x)$, since δ_ε restricted to $F^{\xi}_{f,\varepsilon} \setminus F^{\xi+1}_{f,\varepsilon}$ is continuous. Secondly, assume that infinitely many x_n's are in $X \setminus F^{\xi}_{f,\varepsilon}$. Let all of them be in $X \setminus F^{\xi}_{f,\varepsilon}$. Then by (1) we have $x \notin B(x_n, \delta_\varepsilon(x_n))$ and so, $d(x_n, x) \geq \delta_\varepsilon(x_n) > 0$ for all n. Hence $g = \lim_n \delta_\varepsilon(x_n) = 0 < \delta_\varepsilon(x)$. □

Remark. The authors of [16] observed also that the ε-gauge δ_ε defined in the above proof is B^*_1. The same argument works here, but since we do not want to recall the definition of the class B^*_1, we decided to exclude it from the proof of Theorem 2.1.

We will deduce that the statement defining an equi-Baire 1 family remains valid provided that X and Y are Polish spaces, with a gauge δ_ε being upper semi-continuous.

Theorem 2.2. *Suppose that (X, d_X) and (Y, d_Y) are Polish spaces. Let $\mathcal{F}$ be an equi-Baire 1 family of functions from X to Y. Then for every $\varepsilon > 0$, the respective ε-gauge $\delta_\varepsilon \colon X \to \mathbb{R}_+$ can be chosen upper semi-continuous.*

Proof. Fix $\varepsilon > 0$. By Theorem 1.3 there exists a closed cover $(X_i)_{i\in\mathbb{N}}$ of X such that $\operatorname{diam} f[X_i] \leq \varepsilon$ for all $i \in \mathbb{N}$ and $f \in \mathcal{F}$. Consider a Baire 1 function $g \colon X \to \mathbb{R}$ given by

$$g(x) := \min\{i \in \mathbb{N} \colon x \in X_i\}.$$

Using Theorem 2.1 pick an upper semi-continuous (1/2)-gauge $\delta_\varepsilon \colon X \to \mathbb{R}_+$ for g, and fix $x, x' \in X$ with $d_X(x, x') < \min\{\delta_\varepsilon(x), \delta_\varepsilon(x')\}$. Then $|g(x) - g(x')| \leq \frac{1}{2}$, and it follows that we can find $i \in \mathbb{N}$ such that $x, x' \in X_i$. Hence for each $f \in \mathcal{F}$ we have

$$d_Y(f(x), f(x')) \leq \operatorname{diam} f[X_i] \leq \varepsilon,$$

as desired. □

Corollary 2.3. *Suppose that (X, d_X) and (Y, d_Y) are Polish spaces, and $f \colon X \to Y$ is Baire 1. For every $\varepsilon > 0$ there exists an upper semi-continuous gauge $\delta_\varepsilon \colon X \to \mathbb{R}_+$ such that for all $x, x' \in X$, the condition $d_X(x, x') < \min\{\delta_\varepsilon(x), \delta_\varepsilon(x')\}$ implies $d_Y(f(x), f(x')) \leq \varepsilon$.*

The next problem is inspired by the results of [2].

Problem 2.4. Assume that X is a compact metric space and let $\mathcal{F}$ be an equi-Baire 1 family of (uniformly) bounded functions from X to $\mathbb{R}$. Does, for every $\varepsilon > 0$, there exist an ε-gauge δ_ε as in Definition 1.2 which is upper semi-continuous and of finite oscillation index $\beta(\delta_\varepsilon)$? Recall that there exists a bounded upper semi-continuous function $\delta \colon [0, 1] \to \mathbb{R}_+$ with $\beta(\delta) = \omega$; see [12, Proposition 2].

Theorem 2.2 suggests also the following question.

Question 2.5. What will happen if one requires that the ε-gauge is lower semi-continuous in the statement of Definition 1.2?

An answer follows from [3, Theorem 11] (see also [16, Theorem 1]) where it is proved that every function f described as in the statement of Theorem 1.1, with a lower semi-continuous ε-gauge, must be continuous. So, this restricted version of Definition 1.2 deals only with families of continuous functions. By [1, Example 3.8], an equi-Baire 1 family of continuous functions need not be equi-continuous (our Example 4.2 is slightly stronger, since the limit function is continuous). However, when we use lower semi-continuous gauges and the domain X is compact, the situation is different.

Proposition 2.6. *Assume that (X, d_X), (Y, d_Y) are metric spaces and X is compact. If $\mathcal{F} \subseteq Y^X$ is an equi-Baire 1 family with the respective ε-gauge δ_ε being lower semi-continuous, then $\mathcal{F}$ is equi-continuous.*

Proof. Fix $\varepsilon > 0$ and consider the respective lower semi-continuous gauge δ_ε for $\mathcal{F}$. Then δ_ε attains its minimum in the compact space X. Let $r := \min\{\delta_\varepsilon(t) : t \in X\}$. Let $x, y \in X$ and $d_X(x, y) < r$. Hence $d_X(x, y) < r \le \min\{\delta_\varepsilon(x), \delta_\varepsilon(y)\}$. Since $\mathcal{F}$ is equi-Baire 1, we have $d_Y(f(x), f(y)) < \varepsilon$ for all $f \in \mathcal{F}$. This shows that $\mathcal{F}$ is an equi-continuous family. $\square$

3. Special families of equi-Baire 1 functions

Several examples of equi-Baire 1 families and their properties are given in [1]. One of results is the following (the author assumes that a space X is compact but his proof works in a general case).

Theorem 3.1. *[1] Let (f_n) be a sequence of real-valued Baire 1 functions defined on a metric space X. If (f_n) is uniformly convergent to $f: X \to \mathbb{R}$, then the family $\{f_n : n \in \mathbb{N}\}$ is equi-Baire 1.*

3.1. Convergent sequences

Theorem 3.1 suggests that equi-Baire 1 families can play a similar role to that played by equi-continuous families in the Arzelà-Ascoli theorem. The classical version of this theorem states that any uniformly bounded and equi-continuous sequence of real-valued functions on a compact metric space contains a uniformly convergent subsequence. We will show (see Example 4.2 in the next subsection) that a straightforward counterpart of this result for Baire 1 functions is false.

Remark. It is interesting that, given an equi-Baire 1 family $\mathcal{F}$ of functions between metric spaces X and Y, its closure in the topology of pointwise convergence is also an equi-Baire 1 family; see [14, Proposition 35]. In particular, a pointwise limit of a sequence of functions from $\mathcal{F}$ is Baire 1.

Question 3.2. Let (f_n) be a sequence of (continuous, Baire 1) functions which is pointwise convergent to a Baire 1 function f. Is the family $\{f_n : n \in \mathbb{N}\}$ equi-Baire 1?

We prove below that the answer is positive for a sequence of continuous functions. In the case of Baire 1 functions, the answer is negative which is shown in Example 4.1.

Theorem 3.3. *Let $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$ be a family of continuous functions between Polish spaces (X, d_X) and (Y, d_Y) such that the sequence (f_n) is pointwise convergent on X. Then $\mathcal{F}$ is an equi-Baire 1 family.*

Proof. Fix $\varepsilon > 0$. Since Y is metrizable and separable, there exists a countable cover $(B_i)_{i\in\mathbb{N}}$ of Y by open balls with diameters less than ε. For every $i \in \mathbb{N}$ we can write $B_i = \varphi_i^{-1}[(0,\varepsilon]]$ where $\varphi_i(y) := d_Y(y, Y \setminus B_i)$ is a continuous function. Put $B_{n,i} = \varphi_i^{-1}[[\frac{1}{n},\varepsilon]]$ (we assume $[\frac{1}{n},\varepsilon] := \emptyset$ for $\frac{1}{n} > \varepsilon$). Then $(B_{n,i})_{n\in\mathbb{N}}$ is a sequence of closed subsets in Y such that $B_{n,i} \subseteq \mathrm{int} B_{n+1,i}$ for all $n, i \in \mathbb{N}$. Moreover,

$$B_i = \bigcup_{n=1}^{\infty} B_{n,i} = \bigcup_{n=1}^{\infty} \mathrm{int} B_{n,i}.$$

We will prove that

$$X = \bigcup_{i=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} f_k^{-1}[B_{n,i}].$$

Assume that $f\colon X \to Y$ is the pointwise limit of the sequence (f_n). Fix $x \in X$. Then $f(x) \in B_i$ for some $i \in \mathbb{N}$. There exists n_1 such that $f(x) \in \mathrm{int} B_{n_1,i}$. Since $\lim_{k\to\infty} f_k(x) = f(x)$, there exists $n_2 \geq n_1$ such that $f_k(x) \in \mathrm{int} B_{n_1,i}$ for all $k \geq n_2$. Then for all $k \geq n_2$ we have $f_k(x) \in B_{n_1,i} \subseteq B_{n_2,i}$, since the sequence $(B_{n,i})_{n\in\mathbb{N}}$ increases.

Since every f_k is continuous, the set $F_{n,i} = \bigcap_{k=n}^{\infty} f_k^{-1}[B_{n,i}]$ is closed in X. For all $n, i \in \mathbb{N}$ and $x, y \in F_{n,i}$ we have that $f_k(x) \in B_{n,i}$ and $f_k(y) \in B_{n,i}$, therefore, $\mathrm{diam}\, f_k[F_{n,i}] \leq \varepsilon$ for all $n \geq k$. In other words, for each $\langle n, i\rangle \in \mathbb{N}^2$ there is a finite set $\mathcal{F}' := \{f_k \colon k < n\}$ such that $\mathrm{diam}\, g[F_{n,i}] \leq \varepsilon$ for all $g \in \mathcal{F} \setminus \mathcal{F}'$. Enumerate $\{F_{n,i} \colon i, n \in \mathbb{N}\}$ to a sequence $(A_j)_{j\in\mathbb{N}}$. Then we obtain a countable cover of a space X by closed sets A_j such that for every $j \in \mathbb{N}$ there exists a finite set $\mathcal{F}_j$ of functions from $\mathcal{F}$ with $\mathrm{diam}\, g[A_j] \leq \varepsilon$ for all $g \in \mathcal{F} \setminus \mathcal{F}_j$.

Fix $j \in \mathbb{N}$. Since a finite family $\mathcal{F}_j$ of continuous functions is equi-Baire 1, there exists a cover $(C_{j,m})_{m\in\mathbb{N}}$ of the set A_j such that each $C_{j,m}$ is closed in A_j (and, consequently, in X) and $\mathrm{diam}\, g[C_{j,m}] \leq \varepsilon$ for all $g \in \mathcal{F}_j$. Finally, it remains to enumerate $\{C_{j,m} : j, m \in \mathbb{N}\}$ to a sequence $(X_i)_{i\in\mathbb{N}}$ and apply Theorem 1.3. □

Corollary 3.4. *Let (f_n) be a pointwise convergent sequence of functions with the Baire property where $f_n \colon X \to Y$ and X, Y are Polish spaces. Then there is a G_δ comeager set $E \subseteq X$ such that $\{f_n|_E \colon n \in \mathbb{N}\}$ is an equi-Baire 1 family.*

Proof. It is well-known that f has the Baire property if and only if $f|_E$ is continuous for some comeager set $E \subseteq X$ (see [22, Prop. 3.5.8], [19, Theorem 8.1]). For each n, pick a comeager set $E_n \subseteq X$ such that $f_n|_{E_n}$ is continuous. We may assume that E_n is of type G_δ. Let $E := \bigcap_n E_n$. Then E is a Polish space. The rest follows from Theorem 3.3. □

3.2. Separately equi-Baire 1 functions

Suppose we have a family $\mathcal{F}$ of functions of two variables. If the family $\mathcal{F}$ is equi-Baire 1, then the family of x-sections of functions from $\mathcal{F}$ is again equi-Baire 1 (it follows from Theorem 1.3 (ii), or (iii), or (iv)). We consider a question about the opposite implication.

Let us fix some terminology. Given metric spaces (X, d_X), (Y, d_y), a family $\mathcal{F} \subset Y^X$ is called equi-continuous if, for any $x \in X$ and $\varepsilon > 0$, there exists a number $\delta_\varepsilon(x) > 0$ such that, for all $f \in \mathcal{F}$ and $x' \in X$, the condition $d_X(x, x') < \delta_\varepsilon(x)$ implies $d_Y(f(x), f(x')) < \varepsilon$. If, in this condition, one can find a number $\delta_\varepsilon > 0$ which is good for all $x \in X$, the family $\mathcal{F}$ is called uniformly equi-continuous. The equi-continuity of $\mathcal{F}$ at a point $x \in X$ is meant in a natural manner.

Let X, Y, Z be metric spaces. For $F \colon X \times Y \to Z$ and each $x \in X$ we denote by $F^{\langle x,\cdot\rangle} \colon Y \to Z$ the function defined by the formula $F^{\langle x,\cdot\rangle}(y) = F(x, y)$. Analogously, by $F^{\langle\cdot,y\rangle} \colon X \to Z$ we denote the function $F^{\langle\cdot,y\rangle}(x) = F(x, y)$.

Question 3.5. Suppose that the family of x-sections of functions from $\mathcal{F}$ is equi-continuous (equi-Baire 1) and the family of y-sections of functions from $\mathcal{F}$ is equi-continuous. Is it true that $\mathcal{F}$ is equi-Baire 1? If not, what more should we assume?

The first variant of this question has a simple positive answer under additional assumptions.

Fact 3.6. *Suppose that $\mathcal{F} \subset Z^{X\times Y}$, and the following conditions hold:*

- *for each $x \in X$ the family $\{F^{\langle x,\cdot\rangle} : F \in \mathcal{F}\}$ is uniformly equi-continuous,*
- *for each $y \in Y$ the family $\{F^{\langle \cdot,y\rangle} : F \in \mathcal{F}\}$ is uniformly equi-continuous.*

Then $\mathcal{F}$ is equi-Baire 1.

The proof uses the standard 2ε argument for $\delta_\varepsilon : X \times Y \to \mathbb{R}_+$ given by the formula $\delta_\varepsilon(x,y) := \min\{\delta^x_\varepsilon, \delta^y_\varepsilon\}$.

In general we state the first variant of the above question as an open problem.

Problem 3.7. Suppose that $\mathcal{F} \subset \mathbb{R}^{X\times Y}$ is a family of separately equi-continuous functions defined on the product of Polish spaces X and Y (i.e. $\{F^{\langle x,\cdot\rangle} : F \in \mathcal{F}\}$ is equi-continuous for each $x \in X$ and $\{F^{\langle \cdot,y\rangle} : F \in \mathcal{F}\}$ is equi-continuous for each $y \in Y$). Is $\mathcal{F}$ equi-Baire 1?

The answer to the second variant of Question 3.5 is "no", even if all functions from $\mathcal{F}$ are separately continuous.

Example 3.8. There exists a family $\mathcal{F}$ of separately equi-Baire 1 separately continuous functions such that $\mathcal{F}$ is not equi-Baire 1. Indeed, let $F_n : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a sequence of separately continuous functions which is convergent to $F \notin B_1$. Such a sequence is constructed e.g. in [23, Example 6]. Since for each x (for each y, respectively), $F_n^{\langle x,\cdot\rangle}$ ($F_n^{\langle \cdot,y\rangle}$, resp.) is a sequence of continuous functions convergent pointwise to $F^{\langle x,\cdot\rangle}$ ($F^{\langle \cdot,y\rangle}$, resp.), it is equi-Baire 1 (by Theorem 3.3). But F is not Baire 1 and thus $\{F_n : n \in \mathbb{N}\}$ cannot be equi-Baire 1.

We show how to use separately continuous functions to construct (possibly uncountable) equi-Baire 1 families of functions.

Lemma 3.9. *Suppose that X, Z are metric spaces, Y is a compact topological space, and $\mathcal{F} \subset Z^{X\times Y}$. Fix $x_0 \in X$. Then the following conditions are equivalent:*

(i) *F is jointly continuous at $\langle x_0, y\rangle$ for all $y \in Y$;*
(ii) *$\{F^{\langle \cdot,y\rangle} : y \in Y\}$ is equi-continuous at x_0, and $F^{\langle x_0,\cdot\rangle}$ is continuous on Y.*

Proof. "(i)⇒(ii)": Fix $\varepsilon > 0$. By the compactness of Y there exists a finite family of open sets $U_1, U_2, \ldots, U_N$ such that

- $\{x_0\} \times Y \subset \bigcup_{n=1}^N U_n$, and
- $\operatorname{diam} F[U_n] < \varepsilon$ for all $n \le N$.

We may assume that all U_n's are of the form $B(x_0, \delta_n) \times V_n$, where $V_n \subset Y$ is open. Moreover, we can assume that all δ_n's are equal. Then $d(F^{\langle \cdot,y\rangle}(x), F^{\langle \cdot,y\rangle}(x_0)) < \varepsilon$ for all $x \in B(x_0, \delta_1)$ and $y \in Y$ where d is the metric of Z. Thus $\{F^{\langle \cdot,y\rangle} : y \in Y\}$ is equi-continuous at x_0.

"(ii)⇒(i)": Fix $\varepsilon > 0$. By the continuity of $F^{\langle x_0,\cdot\rangle}$ and the compactness of Y there exists a finite family of open sets $V_1, V_2, \ldots, V_N$ such that

- $Y \subset \bigcup_{n=1}^{N} V_n$, and
- $\operatorname{diam} F^{\langle x_0,\cdot\rangle}[V_n] < \varepsilon$ for all $n \leq N$.

Since $\{F^{\langle\cdot,y\rangle} : y \in Y\}$ is equi-continuous at x_0, there exists $\delta > 0$ such that for all $x \in B(x_0, \delta)$ and $y \in Y$, $\operatorname{dist}(F^{\langle\cdot,y\rangle}(x), F^{\langle\cdot,y\rangle}(x_0)) < \varepsilon$. Then, for all $y \in Y$, there exists an open neighbourhood $U_n := B(x_0, \delta) \times V_n \ni \langle x_0, y\rangle$ such that $\operatorname{diam} F[U_n] < \varepsilon$. Thus F is continuous at $\langle x_0, y\rangle$. □

By the above observation, the well-known result of Namioka [18] for complete metric spaces can be "equivalently" formulated as

Theorem 3.10. *Suppose that X, Z are metric spaces, where X is complete, and Y is a compact topological space. If $F: X \times Y \to Z$ is separately continuous then the family*

$$\{F^{\langle\cdot,y\rangle} : y \in Y\} \text{ is equi-Baire 1.} \tag{2}$$

Proof. The assertion of the theorem of Namioka [18] says that

$$\text{there exists a comeager } G_\delta \text{ set } A \subset X \text{ such that } F \text{ is jointly continuous on } A \times Y. \tag{3}$$

To show (2) we use the characterization from Theorem 1.3(iv). Fix a closed $E \subset X$. By the condition (3) formulated with X replaced by E, there exists an $a \in E$ such that $F|_{E\times Y}$ is jointly continuous at each point of $\{a\} \times Y$. By Lemma 3.9, a is a point of equicontinuity of $\{F^{\langle\cdot,y\rangle}|_E : y \in Y\}$. □

To show the "equivalence" of the above theorem with Namioka's result we have to show that it is an easy consequence of Theorem 3.10. Indeed, using the classical argument one can show that the set

$$A := \{x \in X : \{F^{\langle\cdot,y\rangle} : y \in Y\} \text{ is equicontinuous at } x\}$$

is G_δ. Fix an open $U \subset X$. Then use condition (2) for X replaced by U (which is completely metrizable) and the characterization from Theorem 1.3 (iv), to show the existence of a point of equicontinuity in U. This implies that A is dense.

Remark. For the sequence of functions used in Example 3.8, by (3) there exists a comeager G_δ set $A \subset X$ such that for each n, F_n is continuous (jointly) on $A \times Y$. Then, by Theorem 3.3, the family $\{F_n|_{A\times Y} : n \in \mathbb{N}\}$ is equi-Baire 1 (jointly).

Problem 3.11. Suppose that $F_n : X \times Y \to \mathbb{R}$ is a sequence of separately equi-Baire 1 functions defined on the product of Polish spaces X and Y. Does there exist a comeager G_δ set $A \subset X$ such that $\{F_n|_{A\times Y} : n \in \mathbb{N}\}$ is jointly equi-Baire 1?

3.3. Characteristic functions

Our aim is to check conditions under which a countable family of characteristic functions is equi-Baire 1.

Definition 3.12. (See [13].) A set A in a topological space X is called *resolvable*, if for every nonempty closed set $F \subseteq X$ there exists a relatively open set $U \subseteq F$ such that $U \subseteq A$ or $U \subseteq X \setminus A$.

If X is a hereditarily Baire space (i.e. each closed subset of X is a Baire space), then every F_σ- and G_δ-set is resolvable. The converse statement is true for any perfectly normal paracompact space X. In particular, in complete metric spaces, resolvable sets are exactly F_σ- and G_δ-sets.

Definition 3.13. We say that a family $\mathcal{A}$ of subsets of a topological space X is *equi-resolvable* if for every nonempty closed set $F \subseteq X$ there exists a relatively open set $U \subseteq F$ such that $U \subseteq A$ or $U \subseteq X \setminus A$ for every $A \in \mathcal{A}$.

Lemma 3.14. *If a family $\mathcal{A}$ of subsets of a topological space X is equi-resolvable, then $\bigcup \mathcal{A}'$ is resolvable for every subfamily $\mathcal{A}' \subseteq \mathcal{A}$.*

Proof. Fix any subfamily $\mathcal{A}' \subseteq \mathcal{A}$ and a non-empty closed $F \subset X$. Since $\mathcal{A}$ is equi-resolvable, there exists a relatively open set $U \subseteq F$ such that $U \subseteq A$ or $U \subseteq X \setminus A$ for every $A \in \mathcal{A}$. Consider two cases:

(i) there exists $A \in \mathcal{A}'$ with $U \subseteq A$;
(ii) for all $A \in \mathcal{A}'$, $U \subseteq X \setminus A$.

In case (i) we have $U \subseteq \bigcup \mathcal{A}'$. In case (ii) we have $U \subseteq X \setminus \bigcup \mathcal{A}'$. In both cases, $\bigcup \mathcal{A}'$ is resolvable. □

For $A \subset X$ and $i \in \{0, 1\}$, denote

$$A^i := \begin{cases} A & \text{for } i = 0 \\ X \setminus A & \text{for } i = 1. \end{cases}$$

It is easy to observe that, if $\mathcal{A}$ is equi-resolvable, so is $\{A^i \colon A \in \mathcal{A} \text{ and } i \in \{0, 1\}\}$. Moreover, using Lemma 3.14 and De Morgan's laws, we obtain the next lemma.

Lemma 3.15. *For an equi-resolvable family $\mathcal{A}$ of subsets of a topological space X and any $\alpha \in \{0, 1\}^{\mathcal{A}}$, the set*

$$\bigcap_{A \in \mathcal{A}} A^{\alpha(A)}$$

is resolvable. In particular, if X is Polish, the above set is F_σ.

Theorem 3.16. *Given a topological space X, let $\mathcal{A}$ be a non-empty family of subsets of X and let $\mathcal{F} = \{\chi_A : A \in \mathcal{A}\}$ be the respective family of characteristic functions. Consider the following conditions:*

(i) *for every $\varepsilon > 0$ there exists a closed cover $(X_i)_{i\in\mathbb{N}}$ of X with* $\operatorname{diam} f[X_i] \leq \varepsilon$ *for all $f \in \mathcal{F}$;*
(ii) *$\mathcal{A}$ is equi-resolvable;*
(iii) *the family $\mathcal{B} := \{\bigcap_{A\in\mathcal{A}} A^{\alpha(A)} \colon \alpha \in \{0, 1\}^{\mathcal{A}}\}$ is countable and contains only F_σ sets.*

Then: (i) ⇒ (ii) *if X is hereditarily Baire;* (ii) ⇒ (iii) *if X is perfectly normal paracompact space:* (iii) ⇒ (i).

In particular, if X is a Polish space, all these conditions are equivalent.

Proof. We start with two observations about the family $\mathcal{B}$ defined in (iii):

- elements of $\mathcal{B}$ are pairwise disjoint;

- $\bigcup \mathcal{B} = X$,

i.e. $\mathcal{B}$ forms a partition of X.

"(i)⇒(ii)": Assume that X is hereditarily Baire. Fix a closed set $F \subseteq X$ and notice that F is a Baire space. There exists a closed cover $(X_i)_{i\in\mathbb{N}}$ of X such that $\operatorname{diam} f[X_i] = 0$ for all $f \in \mathcal{F}$. Since F is Baire, there is N such that $U = \operatorname{int}_F(X_N \cap F) \neq \emptyset$. Then $\operatorname{diam} f[U] = 0$ for every $f \in \mathcal{F}$, which implies that $U \subseteq A$ or $U \subseteq X \setminus A$ for every $A \in \mathcal{A}$.

"(ii)⇒(iii)": Lemma 3.15 implies that each element of $\mathcal{B}$ is resolvable, and so it is an F_σ and G_δ subset of X. It is enough to show that $\mathcal{B}$ is countable. Assume to the contrary that $\mathcal{B}$ is uncountable. Let B be a transversal of $\mathcal{B}$, i.e. cardinality of $B \cap A$ is equal to 1 for all $A \in \mathcal{B}$ (it is possible since $\mathcal{B}$ is a partition of X). Then B is also uncountable. Let $C \subseteq X$ be the closure of B. Since C is uncountable, it contains a non-empty perfect set $F \subseteq C$. Since $\mathcal{A}$ is equi-resolvable, there exists an open set $U \subset X$ such that $U \cap F \neq \emptyset$ and for each $A \in \mathcal{A}$,

$$\text{either } U \cap F \subseteq A \text{ or } U \cap F \subseteq X \setminus A. \tag{4}$$

But $U \cap B \neq \emptyset$, and so (since F is dense in itself), $U \cap B$ has at least two elements, say b_1, b_2. Thus there exist $\alpha_1, \alpha_2 \in \{0,1\}^{\mathcal{A}}$ with

$$b_1 \in \bigcap_{A\in\mathcal{A}} A^{\alpha_1(A)} \neq \bigcap_{A\in\mathcal{A}} A^{\alpha_2(A)} \ni b_2,$$

and so

$$U \cap B \cap \bigcap_{A\in\mathcal{A}} A^{\alpha_1(A)} \neq \emptyset \neq U \cap B \cap \bigcap_{A\in\mathcal{A}} A^{\alpha_2(A)}.$$

Moreover, for any $A \in \mathcal{A}$ with $\alpha_1(A) \neq \alpha_2(A)$,

$$\chi_A\left[U \cap B \cap \bigcap_{A\in\mathcal{A}} A^{\alpha_1(A)}\right] \neq \chi_A\left[U \cap B \cap \bigcap_{A\in\mathcal{A}} A^{\alpha_2(A)}\right].$$

This contradicts with (4).

"(iii)⇒(i)": Fix $\varepsilon \in (0,1)$. Since $\mathcal{B}$ is countable, $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$. Each of these sets is F_σ in X. Therefore, for every $n \in \mathbb{N}$ there exists a sequence of closed sets $(F_{nm})_{m\in\mathbb{N}}$ such that

$$B_n = \bigcup_{m\in\mathbb{N}} F_{nm}.$$

Enumerate the double sequence $(F_{nm})_{n,m\in\mathbb{N}}$ into $(X_n)_{n\in\mathbb{N}}$. Since $\mathcal{B}$ is a partition of X, we have $X = \bigcup_{n\in\mathbb{N}} X_n$. Moreover, $\operatorname{diam} f[B_n] = 0$ for all $n \in \mathbb{N}$ and $f \in \mathcal{F}$, and it follows that $\operatorname{diam} f[X_n] = 0$ for all f's and n's. □

Corollary 3.17. *Consider a non-empty family $\mathcal{A}$ of pairwise disjoint subsets of X. Then the family $\{\chi_A : A \in \mathcal{A}\}$ is equi-Baire 1 if and only if*

- *$\mathcal{A}$ is countable, and*
- *each element of $\mathcal{A}$ is F_σ, and*
- *$\bigcup \mathcal{A}$ is G_δ.*

Proof. If elements of $\mathcal{A}$ are pairwise disjoint then

$$\left\{ \bigcap_{A\in\mathcal{A}} A^{\alpha(A)} \colon \alpha \in \{0,1\}^{\mathcal{A}} \right\} = \begin{cases} \mathcal{A} \cup \{X \setminus \bigcup \mathcal{A}\} & \text{if } \mathcal{A} \text{ has exactly one element;} \\ \mathcal{A} \cup \{\emptyset, X \setminus \bigcup \mathcal{A}\} & \text{if } \mathcal{A} \text{ has at least two elements.} \end{cases}$$

The rest follows from Theorem 3.16 (iii). □

As an immediate consequence of the previous corollary, we obtain the following.

Corollary 3.18. *Consider a countable set $A = \{a_n \colon n \in \mathbb{N}\} \subseteq \mathbb{R}$. Then the family $\{\chi_{\{a_n\}} \colon n \in \mathbb{N}\}$ is equi-Baire 1 if and only if A is G_δ.*

4. Applications

We give two examples as an application of Corollary 3.18. Example 4.1 answers the Baire 1 part of Question 3.2 in the negative. Example 4.2 shows that the generalization of the Arzelà-Ascoli theorem on the class of Baire 1 functions is false.

Example 4.1. Fix a one-to-one enumeration q_n, $n \in \mathbb{N}$, of all rational numbers and let $f_n := \chi_{\{q_n\}}$ for $n \in \mathbb{N}$. Clearly, the sequence (f_n) is pointwise convergent to the zero function. It is known that $\mathbb{Q}$ is not a G_δ set. Hence $\{f_n \colon n \in \mathbb{N}\}$ is not an equi-Baire 1 family.

We can also give a direct proof of this fact. Indeed, suppose that there is a (1/2)-gauge $\delta \colon \mathbb{R} \to \mathbb{R}_+$ which witnesses that $\{f_n \colon n \in \mathbb{N}\}$ is equi-Baire 1. Let $X_n := \{x \in \mathbb{R} \colon \delta(x) > 1/n\}$ for $n \in \mathbb{N}$. Then $\mathbb{R} = \bigcup_n X_n$ and, by the Baire category theorem, pick a set X_m whose closure has nonempty interior. So, pick a ball $B(q_n, r)$ contained in the closure of X_m with $r < \min\{1/m, \delta(q_n)\}$. Take a point x in $B(q_n, r) \cap (X_m \setminus \{q_n\})$. Then $|x - q_n| < r < 1/m < \delta(x)$ and $r < \delta(q_n)$. Hence $|f_n(x) - f_n(q_n)| < 1/2$. On the other hand, $|f_n(x) - f_n(q_n)| = 1$. Contradiction.

Example 4.2. Let $X := [0, 1]$. The set $A := \{0\} \cup \{1/n : n \geq 2\} \subseteq X$ is closed, hence G_δ. By Corollary 3.18, the family $\mathcal{F}$ of characteristic functions of singletons contained in A is equi-Baire 1. Clearly, the functions in $\mathcal{F}$ are uniformly bounded. Let $f_1 := \chi_{\{0\}}$ and $f_n := \chi_{\{1/n\}}$ for $n \geq 2$. Consider a subsequence (f_{k_n}) of (f_n). Then for any $N \in \mathbb{N}$ there are $m, n \geq N$ such that

$$\sup_{x\in[0,1]} |f_{k_n}(x) - f_{k_m}(x)| = 1.$$

Hence (f_{k_n}) cannot be uniformly convergent. We can modify this example to make the functions f_n continuous. Indeed, for $n \geq 2$ and $x \in [0, 1]$, let

$$\overline{f_n}(x) := n(n+1)\text{dist}\left(x, \left[0, \frac{1}{n} - \frac{1}{n+1}\right] \cup \left[\frac{1}{n} + \frac{1}{n+1}, 1\right]\right).$$

These functions are continuous and bounded by 1. The family $\{\overline{f_n} \colon n \geq 2\}$ is equi-Baire 1 since $(\overline{f_n})$ converges pointwise to the zero function, so we can use Theorem 3.3. As above, we argue that there is no uniformly convergent subsequence of $(\overline{f_n})$.

Comparing Theorem 3.1 with Example 4.2, one can raise the following question.

Question 4.3. Given a uniformly bounded sequence (f_n) of real-valued Baire 1 functions defined on a Polish space X, assume that the family $\{f_n \colon n \in \mathbb{N}\}$ is equi-Baire 1. What more should be assumed to guarantee that (f_n) contains a subsequence which is uniformly convergent on X? We can also ask whether the above sequence (f_n) contains a subsequence that is uniformly convergent on a large set in the Baire category sense.

As to the second part of this question, if we consider its measure counterpart, the answer is positive provided that a finite Borel measure μ is given on X. Indeed, it suffices to apply the Egorov theorem (see [19, Theorem 8.3]). Then the whole sequence is uniformly convergent on a set of measure sufficiently close to $\mu(X)$. It is known (see [19]) that the Baire category analogue of the Egorov theorem is false. This also gives the negative answer to our question.

Example 4.4. We will adapt the example from [19, Section 8]. Define the sequence $(\varphi_n)_{n\geq 0}$ of tent-like continuous functions $\varphi_n : \mathbb{R} \to \mathbb{R}$, $n \geq 0$, as follows. Firstly, let

$$\varphi_0(x) := 2x \quad \text{for } x \in [0, 1/2] \quad \text{and} \quad \varphi_0(x) := 2 - 2x \quad \text{for} \quad x \in [1/2, 1].$$

Secondly, let $n \geq 1$ and put $\varphi_n(x) := \varphi_0(2^n x)$ for $x \in [0, 1/2^n]$. Finally, let $\varphi_n(x) := 0$ for all $n \geq 0$, whenever $x < 0$ or $x > 1/2^n$. Enumerate all rationals in a one-to-one sequence (q_i). For $n \geq 0$, let

$$f_n(x) := \sum_{i=1}^{\infty} 2^{-i} \varphi_n(x - q_i) \quad \text{if} \quad x \in \mathbb{R}.$$

Then every function f_n is continuous since the above series is uniformly convergent on $\mathbb{R}$. Additionally, $f_n \to 0$ pointwise on $\mathbb{R}$. If an interval (a, b) is fixed, there is no subsequence (f_{k_n}) of (f_n) that can be uniformly convergent on (a, b) since, if $q_i \in (a, b)$, we have $\sup_{x\in(a,b)} f_{k_n}(x) \geq 1/2^i$ for sufficiently large n. Moreover, any set on which (f_{k_n}) is uniformly convergent, must be nowhere dense (see [19]). Observe that $\{f_n : n \in \mathbb{N}\}$ is an equi-Baire 1 family by Theorem 3.3.

Inspired by a version of the Arzelà-Ascoli theorem for pointwise convergence [21, Theorem 2.3.14], we can, thanks to Theorem 1.3, prove the analogous version for Baire 1 functions.

Proposition 4.5. *Let X be a separable metric space and let (f_n) be a pointwise bounded sequence of functions $f_n : X \to \mathbb{R}$, $n \in \mathbb{N}$, that form an equi-Baire 1 family. Then there exists a subsequence (f_{k_n}) which is pointwise convergent on X to a Baire 1 function.*

Proof. By Theorem 1.3 consider a finer separable topology on X which makes $\mathcal{F}$ equi-continuous. From now on, we will use this topology on X and assume that it is metrizable by a metric d. To end the proof, it suffices to apply the above mentioned version of the Arzelà-Ascoli theorem dealing with pointwise convergence. We recall briefly its proof for the reader's convenience.

Fix a countable dense set $\{q_r : r \in \mathbb{N}\} \subseteq X$. Since (f_n) is pointwise bounded, we can, by the Bolzano-Weierstrass theorem and the standard diagonal trick, choose a subsequence (f_{k_n}) that is convergent at every point q_r.

Let $\varepsilon > 0$ and $x \in X$. Since the family $\{f_n : n \in \mathbb{N}\}$ is equi-continuous, consider $\delta_x > 0$ such that $|f_n(x') - f_n(x)| < \varepsilon/3$ for all $n \in \mathbb{N}$ whenever $x' \in X$ and $d(x', x) < \delta_x$. Choose $q_r \in B(x, \delta_x)$. Since $(f_{k_n}(q_r))_{n\in\mathbb{N}}$ is convergent, it is a Cauchy sequence, so pick $N \in \mathbb{N}$ such that $|f_{k_m}(q_r) - f_{k_p}(q_r)| < \varepsilon/3$ for all $m, p \geq N$. Then for all $m, p \geq N$ we have

$$|f_{k_m}(x) - f_{k_p}(x)| \leq |f_{k_m}(x) - f_{k_m}(q_r)| + |f_{k_m}(q_r) - f_{k_p}(q_r)| + |f_{k_p}(q_r) - f_{k_p}(x)| < \varepsilon.$$

Hence $(f_{k_n}(x))$ is a Cauchy sequence which implies its convergence.

Finally, note that the pointwise limit of the sequence (f_{k_n}) of equi-Baire 1 functions is Baire 1 by [14, Proposition 36]. □

The above proposition can be derived from the deep result by Bourgain, Fremlin and Talagrand [4]. Indeed, if the family $\mathcal{F} := \{f_n : n \in \mathbb{N}\}$ is equi-Baire 1, then its closure in the pointwise topology is also equi-Baire 1 by [14, Proposition 36], hence it is contained in the class of Baire 1 functions. If $\mathcal{F}$ is pointwise bounded, this closure is compact in the topology of pointwise convergence. But such a compact set, if it is a subset of Baire 1 functions, is also sequentially compact by the result of [4].

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