A CHARACTERIZATION OF THE UNIFORM CONVERGENCE POINTS SET OF SOME CONVERGENT SEQUENCE OF FUNCTIONS

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Abstract. We characterize the uniform convergence points set of a pointwisely convergent sequence of real-valued functions defined on a perfectly normal space. We prove that if X is a perfectly normal space which can be covered by a disjoint sequence of dense subsets and $A\subseteq X$, then A is the set of points of the uniform convergence for some convergent sequence $(f_n)_{n\in\omega}$ of functions $f_n:X\to\mathbb{R}$ if and only if A is G_δ-set which contains all isolated points of X. This result generalizes a theorem of Ján Borsík published in 2019.

Dedicated to the memory of Ján Borsík

1. Introduction

Let X be a topological space, $(Y,|\cdot-\cdot|)$ be a metric space; $B(a,r)$ and $B[a,r]$ be an open and a closed ball in Y with a center $a\in Y$ and a radius $r>0$, respectively. By ∂A we denote a boundary of a set A.

Let $\mathscr{F}=(f_n)_{n\in\omega}$ be a sequence of functions $f_n:X\to Y$. We denote $PC(\mathscr{F})$ the set of all points $x\in X$ such that the sequence $(f_n(x))_{n\in\omega}$ is convergent in Y. Therefore, we define the limit function $f(x)$ by the rule $f(x)=\lim_{n\to\infty}f_n(x)$ for all $x\in PC(\mathscr{F})$. Let us observe that the set $PC(\mathscr{F})$ can be represented in the form

$$PC(\mathscr{F})=\bigcap_{k\in\omega}\bigcup_{n\in\omega}\bigcap_{m\in\omega}f_{n+m}^{-1}(B[f(x),\tfrac{1}{k+1}]). \tag{1.1}$$

If every function f_n is continuous, then the set $PC(\mathscr{F})$ is $F_{\sigma\delta}$ in X. Hans Hahn [6] and Wacław Sierpiński [11] proved independently that the converse proposition is true for metrizable X and $Y=\mathbb{R}$, that is, for every $F_{\sigma\delta}$-subset A of a metrizable space X there exists a sequence $\mathscr{F}$ of real-valued continuous functions $f_n:X\to\mathbb{R}$ such that $A=PC(\mathscr{F})$.

After appearance of this theorem many results were obtained in similar directions: other types of convergence and other classes of functions were considered (see, for instance, [1, 4, 7, 8, 9, 10, 12, 13, 14]). Ján Borsík studied in [1], in particular, the uniform convergence points set of a (convergent pointwisely) sequence of functions.

Definition 1. A sequence $(f_n)_{n\in\omega}$ of functions $f_n:X\to Y$ between a topological space X and a metric space $(Y,|\cdot-\cdot|)$ is *uniformly Cauchy at a point* $x_0\in X$, if for every $\varepsilon>0$ there exist a neighborhood U of x_0 and a number $n_0\in\omega$ such that $|f_n(x)-f_m(x)|<\varepsilon$ for all $n,m\ge n_0$ and $x\in U$.

Let $UC(\mathscr{F})$ be a set of all points with the uniform Cauchy property for a sequence $\mathscr{F}=(f_n)_{n\in\omega}$. It is easy to see that if $(f_n)_{n\in\omega}$ is convergent pointwisely on X to a function $f:X\to Y$, then

$$UC(\mathscr{F})=\{x\in X:\forall\varepsilon>0\ \exists U\ni x_0\ \exists n_0\ \forall n\ge n_0\ |f_n(y)-f(y)|<\varepsilon\ \forall y\in U\}. \tag{1.2}$$

Moreover, in this case $UC(\mathscr{F})$ is the set of all points of the uniform convergence of $\mathscr{F}$.

Borsík proved the following result.

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Theorem A (Borsík, [1])**.** *Let X be a metric space and $A \subseteq X$. Then $A = UC(\mathscr{F})$ for some convergent sequence $\mathscr{F} = (f_n)_{n\in\omega}$ of functions $f_n : X \to \mathbb{R}$ if and only if A is G_δ and contains all isolated points of X.*

This short note is inspirited by the above mentioned paper of Ján Borsík. We generalize his theorem on a wider class of topological spaces.

Definition 2. A topological space X is *ω-resolvable* if there exists a partition $\{X_n : n \in \omega\}$ of X by dense subsets.

For crowded spaces (i.e., spaces without isolated points) the class of all ω-resolvable spaces includes all metrizable spaces, Hausdorff countably compact spaces, arcwise connected spaces, etc. [2]

The main result of our note is the following theorem.

Theorem 1. *Let X be a perfectly normal ω-resolvable space and $A \subseteq X$. The following conditions are equivalent:*

(i) A is a set of all points of uniform convergence for some convergent sequence $(f_n)_{n\in\omega}$ of functions $f_n : X \to \mathbb{R}$;

(ii) A is a G_δ-set which contains all isolated points of X.

2. Proof of Theorem 1

The implication *(i)* $\Rightarrow$ *(ii)* follows immediately from the equality (1.2).

(ii) $\Rightarrow$ *(i)*. Let $(G_n)_{n\in\omega}$ be a sequence of open sets in X such that $A = \bigcap_{n\in\omega} G_n$, $G_{n+1} \subseteq G_n$ for every $n \in \omega$ and let $G_0 = X$.

Since X is perfectly normal, for every $n \in \mathbb{N}$ there exist continuous functions $\varphi_n, \psi_n : X \to [0, 1]$ such that $\varphi_n^{-1}(0) = \overline{G_n}$ and $\psi_n^{-1}(0) = X \setminus G_n$. Then every function $\alpha_n : X \to [-1, 1]$ defined by the formula $\alpha_n = \varphi_n - \psi_n$ has the following properties:

$$\begin{aligned} &\alpha_n(x) < 0 && \forall x \in G_n, \\ &\alpha_n(x) = 0 && \forall x \in \partial G_n, \\ &\alpha_n(x) > 0 && \forall x \in X \setminus \overline{G_n}. \end{aligned}$$

We consider functions $\beta_n : X \to [-1, 1]$ defined by the rule

$$\beta_n(x) = \max_{i\le n} \alpha_i(x)$$

for all $x \in X$. Then we claim that

$$\partial G_n = \beta_n^{-1}(0)$$

for every $n \in \mathbb{N}$. We need to prove $\beta_n(x) = 0 \ \Leftrightarrow\ \alpha_n(x) = 0$ for every $x \in X$. Assume that $\beta_n(x) = 0$. Then $\alpha_i(x) \le 0$ for all $i \le n$ and $\alpha_k(x) = 0$ for some $k \le n$. Then $x \in \overline{G_i}$ for all $i \le n$. If $x \notin \partial G_n$, then $x \in G_n \subseteq G_i$ for all $i \le n$, consequently, $\alpha_k(x) < 0$, a contradiction. Hence, $x \in \partial G_n$ and $\alpha_n(x) = 0$. Conversely, if $\alpha_n(x) = 0$, then $x \in \partial G_n \subseteq \overline{G_n} \subseteq \overline{G_i}$ for all $i \le n$. In consequence, $\alpha_i(x) \le 0$ for all $i < n$ and $\beta_n(x) = 0$.

Now we put

$$\gamma_n(x) = \min_{i\le n} |\beta_i(x)|$$

and notice that

$$\gamma_n^{-1}(0) = \bigcup_{i\le n} \partial G_i.$$

Finally, let

$$\delta_n(x) = \begin{cases} 0, & x \in \overline{G_n}, \\ \gamma_n(x), & x \in X \setminus \overline{G_n}. \end{cases}$$

Obviously, the functions β_n, γ_n and δ_n are continuous and $\delta_n \leq \gamma_n$.

We put $U_{k,0} = F_{k,0} = \emptyset$, $k \in \mathbb{N}$. For all $k, n \in \mathbb{N}$ we define

$$U_{k,n} = \gamma_n^{-1}([0, \tfrac{1}{k})) \quad \text{and} \quad F_{k,n} = \delta_n^{-1}([\tfrac{1}{k}, 1]).$$

The sets $U_{k,n}$ and $F_{k,n}$ satisfy the following conditions:

(A) $U_{k,n}$ is open and $F_{k,n}$ is closed in X,

(B) $\overline{U_{k+1,n}} \subseteq U_{k,n} \subseteq U_{k,n+1}$,

(C) $F_{k,n} \subseteq F_{k+1,n} \cap F_{k,n+1}$,

(D) $\delta_n^{-1}((0,1]) = \bigcup_{k\in\mathbb{N}} F_{k,n} = X \setminus \left(\bigcup_{i=1}^n \partial G_i \cup G_n\right)$,

(E) $\gamma_n^{-1}(0) = \bigcup_{i=1}^n \partial G_i = \bigcap_{k\in\mathbb{N}} U_{k,n}$,

(F) $U_{k,n} \cap F_{k,n} = \emptyset$

for all $k, n \in \mathbb{N}$.

Moreover, the sets G_n satisfy the property

(G) $(\partial G_n \setminus \partial G_{n-1}) \cap \partial G_i = \emptyset$, $i < n$, $n \in \mathbb{N}$.

Since the most of properties are evident, we prove only (C) and (G).

(C). It is enough to prove that $F_{k,n} \subseteq F_{k,n+1}$. Fix $x \in F_{k,n}$ for some $k, n \in \mathbb{N}$. Then $\delta_n(x) \geq \frac{1}{k}$ and, in consequence, $x \notin \overline{G_n}$. Therefore, $\alpha_n(x) > 0$ and $\alpha_{n+1}(x) > 0$. Hence, $\beta_n(x) > 0$ and $\beta_{n+1}(x) > 0$. Since $\gamma_n(x) = \delta_n(x) \geq \frac{1}{k}$, the inequality $|\beta_i(x)| \geq \frac{1}{k}$ holds for all $i \leq n$. In particular, $|\beta_n(x)| = \beta_n(x) \geq \frac{1}{k}$. Then $|\beta_{n+1}(x)| = \beta_{n+1}(x) \geq \beta_n(x) \geq \frac{1}{k}$. Thus, $\delta_{n+1}(x) = \gamma_{n+1}(x) = \min\limits_{i\leq n+1} |\beta_i(x)| \geq \frac{1}{k}$ and $x \in F_{k,n+1}$.

(G). Fix $x \in \partial G_n \setminus \partial G_{n-1}$. Since $x \in \partial G_n$, $x \in \overline{G_n} \setminus G_n$. Then $x \in \overline{G_{n-1}}$, because the sequence $(G_n)_{n\in\omega}$ is decreasing. Moreover, $x \notin \partial G_{n-1}$ and therefore $x \in G_{n-1}$. Again, since $(G_n)_{n\in\omega}$ decreases, $x \in G_i$ for all $i < n$. Hence, $x \notin \partial G_i$, $i < n$.

Since $X \setminus \overline{A}$ is an open subset of the ω-resolvable space X, it is ω-resolvable also. Hence, there exists a sequence $(B_k)_{k\in\omega}$ of mutually disjoint subsets of $X \setminus \overline{A}$ such that

$$X \setminus \overline{A} = \bigcup_{k\in\omega} B_k$$

and each set B_k in dense in $X \setminus \overline{A}$.

For every $x \in X$ we put

$$f(x) = \begin{cases} \frac{1}{n}, & x \in \partial G_n \setminus \partial G_{n-1} \text{ for some } n \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

Notice that f is correctly defined because of property (G).

Now let

$$C_{k,n} = (U_{k,n} \setminus U_{k,n-1}) \cup (B_k \cap (F_{k,n} \setminus F_{k,n-1}))$$

and

$$f_k(x) = \begin{cases} \frac{1}{n}, & \text{if} \quad x \in C_{k,n} \text{ for some } n \geq k, \\ 0, & \text{otherwise.} \end{cases}$$

In order to show that $(f_k)_{k\in\omega}$ converges to f pointwisely on X we fix $x \in X$.

If $x \in \bigcup_{n\in\omega} \partial G_n$, then we put $N = \min\{n \in \omega : x \in \partial G_n\}$. Therefore, since $\partial G_0 = \emptyset$, property (G) implies that $N \in \mathbb{N}$ and $x \in \partial G_N \setminus \bigcup_{i<N} \partial G_i = \partial G_N \setminus \partial G_{N-1}$. Hence, $f(x) = \frac{1}{N}$. Then by (F) we conclude that $x \in \partial G_N \subseteq \bigcap_{k\in\mathbb{N}} U_{k,N}$ and $x \notin \bigcap_{i<N-1} \partial G_i = \bigcap_{k\in\mathbb{N}} U_{k,N-1}$. In consequence, taking into account (B) we conclude that there exists $K \in \omega$ such that $x \in U_{k,N} \setminus U_{k,N-1}$ for all $k \geq K$. Therefore, $f_k(x) = \frac{1}{N} = f(x)$ for all $k \geq K$. Hence, $\lim_{k\to\infty} f_k(x) = f(x)$.

If $x \notin \bigcup_{n\in\omega} \partial G_n$, then $f(x) = 0$. Let $\varepsilon > 0$ and $n \in \mathbb{N}$ be such that $\frac{1}{n} < \varepsilon$. Using (F) we conclude that $x \notin \bigcup_{i\leq n} \partial G_i = \bigcap_{k\in\mathbb{N}} U_{k,n}$. Then, taking into account (B), we obtain that there is $K_1 \in \mathbb{N}$ such that $x \notin U_{k,n}$ for all $k \geq K_1$. But $x \in \bigcup_{k\in\omega} B_k$ and the sets B_k are disjoint, so there is $K > K_1$ such that $x \notin \bigcup_{k\geq K} B_k$. Therefore, we have $x \notin \bigcup_{k\geq K}(U_{k,n} \cup B_k)$.

Assume that $k \geq K$. Consider the case $x \in U_{k,k}$. Then we choose the minimal number $m \leq k$ such that $x \in U_{k,m}$. Then $m > n$ (indeed, if $m \leq n$, then $x \in U_{k,m} \subseteq U_{k,n}$, a contradiction). In particular, $m > 1$ and $x \in U_{k,m} \setminus U_{k,m-1} \subseteq C_{k,m}$. Therefore, $f_k(x) = \frac{1}{m} < \frac{1}{n} < \varepsilon$. Now we consider the case $x \notin U_{k,k}$, then (B) implies that $x \notin U_{k,n}$ for any $n \leq k$. But $x \notin B_k$. Hence, $x \notin C_{k,n}$ for every $n \geq k$. Thus, $f_k(x) = 0 < \varepsilon$.

Now we prove that $A = UC(\mathscr{F})$ for $\mathscr{F} = (f_n)_{n\in\omega}$. Fix $x \in A$, $\varepsilon > 0$ and let $n_0 \in \mathbb{N}$ be such that $\frac{1}{n_0} < \varepsilon$. Since $x \in A$, we conclude that $x \in G_i$ for every i and then $x \notin \bigcup_{i\leq n_0} \partial G_i = \bigcap_{k\in\mathbb{N}} U_{k,n_0}$. Then there exists $k_0 \geq n_0$ with $x \notin U_{k_0-1,n_0}$. Therefore, property (B) implies that $x \notin \overline{U_{k,n_0}}$ for all $k \geq k_0$.

We consider an open neighborhood

$$U = G_{n_0} \setminus \overline{U_{k_0,n_0}}$$

of x in X. Take an arbitrary $u \in U$ and $k \geq k_0$. Since $u \in G_{n_0}$, we have that $u \in G_i$ and then $u \notin \partial G_i$ for any $i \leq n_0$. Therefore,

$$f(u) \in [0, \tfrac{1}{n_0}). \tag{2.2}$$

Let us observe that (B) and (C) imply that $C_{k,n} \subseteq U_{k,k} \cup (B_k \cap F_{k,k})$ for all $k \geq n$. Therefore, if $u \notin U_{k,k} \cup (F_{k,k} \cap B_k)$, then $f_k(u) = f(u) = 0$.

Let us consider the case $u \in U_{k,k}$. Then we take the minimal $i \leq k$ such that $x \in U_{k,i}$. Notice that $i > n_0$ (indeed, if $i \leq n_0$, then (B) implies that $u \in U_{k,i} \subseteq U_{k,n_0} \subseteq U_{k_0,n_0}$, a contradiction). In particular, $i > 1$ and then $u \in U_{k,i} \setminus U_{k,i-1} \subseteq C_{k,i}$. Thus, $f_k(u) = \frac{1}{i} < \frac{1}{n_0}$.

In the case $u \in F_{k,k}$ we choose the minimal $j \leq k$ with $u \in F_{k,j}$. Observe that $j > n_0$ (indeed, if $j \leq n_0$, then $u \in F_{k,j} \subseteq F_{k,n_0}$, and so (D) implies $u \notin G_{n_0}$, which is impossible). In particular, $j > 1$ and $u \in B_k \cap (F_{k,j} \setminus F_{k,j-1}) \subseteq C_{k,j}$. Therefore, $f_k(x) = \frac{1}{j} < \frac{1}{n_0}$.

Thus, we proved that in any case $f_k(u) \in [0, \frac{1}{n_0})$. Hence,

$$|f(u) - f_k(u)| \leq \frac{1}{n_0} < \varepsilon.$$

Therefore, $A \subseteq UC(\mathscr{F})$.

Now we prove that $UC(\mathscr{F}) \subseteq A$. In order to do this we fix $x \notin A$ and show that $x \notin UC(\mathscr{F})$ in this case. Let $n_0 = \max\{n \in \omega : x \in G_n\}$, $\varepsilon = \frac{1}{n_0+1} - \frac{1}{n_0+2}$, U be an open neighborhood of x and let $k_0 \in \omega$. Notice that $x \in G_{n_0} \setminus G_{n_0+1}$.

Consider the case $x \in \overline{G_{n_0+1}}$. Then $x \in \overline{G_{n_0+1}} \setminus G_{n_0+1} = \partial G_{n_0+1}$. Therefore, (E) implies that $x \in \partial G_{n_0+1} \subseteq \bigcap_{k\in\mathbb{N}} U_{k,n_0+1}$. On the other hand, $x \in G_{n_0}$. So, $x \notin \partial G_i$ for any $i \leq n_0$. Using

(E) we conclude that $x \notin \bigcup_{i\le n_0} \partial G_i = \bigcap_{k\in\mathbb{N}} U_{k,n_0}$. Then, there exists $k_1 \in \mathbb{N}$ such that $x \notin U_{k_1,n_0}$. Therefore, (B) implies that $x \notin \overline{U_{k,n_0}}$ for all $k \ge k_1$. Hence, there exists $k > \max\{k_0, n_0\}$ such that $x \in U_{k,n_0+1} \setminus \overline{U_{k,n_0}}$.

Since the set $\bigcup_{n\le n_0+1} \partial G_n$ is nowhere dense in X, there is a nonempty open set V such that $V \subseteq U \cap (U_{k,n_0+1} \setminus \overline{U_{k,n_0}}) \setminus \bigcup_{n\le n_0+1} \partial G_n$. Take $u \in V$. Then $f(u) \in [0, \frac{1}{n_0+2}]$, because $u \notin \bigcup_{n\le n_0+1} \partial G_n$, and $f_k(u) = \frac{1}{n_0+1}$, since $u \in U_{k,n_0+1} \setminus U_{k,n_0}$. Therefore,

$$|f(u) - f_k(u)| \ge \tfrac{1}{n_0+1} - \tfrac{1}{n_0+2} = \varepsilon.$$

Now we assume that $x \notin \overline{G_{n_0+1}}$. Then $x \in G_{n_0} \notin \overline{G_{n_0+1}}$. Therefore, $x \notin \bigcup_{i\le n_0+1} \partial G_i = \gamma_{n_0+1}^{-1}(0)$. Consequently, $\gamma_{n_0+1}(x) > 0$. So, there exists $m_0 \in \mathbb{N}$ such that $\frac{1}{m_0} < \gamma_{n_0+1}(x)$. But $x \notin \overline{G_{n_0+1}}$. Therefore, $\delta_{n_0+1}(x) = \gamma_{n_0+1}(x) > \frac{1}{m_0}$. Hence, $x \in \operatorname{int} F_{m_0,n_0+1}$.

By property (C), there exists a number $k > \max\{k_0, m_0, n_0\}$ such that $x \in \operatorname{int} F_{k,n_0+1}$. Since $x \in G_{n_0}$, $x \notin F_{k,n_0}$. Then the set

$$G = (U \setminus \overline{G_{n_0+1}}) \cap (\operatorname{int} F_{k,n_0+1} \setminus F_{k,n_0})$$

is an open neighborhood of x. Since $\bigcup_{n\le n_0+1} \partial G_n$ is nowhere dense in X and B_k is dense in $X \setminus \overline{A}$, there exists a point $v \in X$ such that

$$v \in (G \setminus \bigcup_{n\le n_0+1} \partial G_n) \cap B_k.$$

Then $f_k(v) = \frac{1}{n_0+1}$, since $v \in C_{k,n_0+1}$, and $f(v) \in [0, \frac{1}{n_0+2}]$, because $v \notin \bigcup_{n\le n_0+1} \partial G_n$. Hence,

$$|f(v) - f_k(v)| \ge \tfrac{1}{n_0+1} - \tfrac{1}{n_0+2} = \varepsilon.$$

Therefore, $A = UC(\mathscr{F})$. □

Remark 1. Actually, we use in the proof only the fact that the boundary of every open set in a topological space X is a functionally closed set. It is find out that this is a characterization of perfectly normal spaces. Moreover, the following conditions are equivalent:

(i) X is a perfectly normal space;

(ii) every closed nowhere dense subset of X is functionally closed.

Evidently, (i) ⇒ (ii). In order to prove (ii) ⇒ (i) we take a closed set $F \subseteq X$. Since $\partial F = F \setminus \operatorname{int} F$ is closed and nowhere dense, there exists a continuous function $g : X \to [0, 1]$ such that $\partial F = g^{-1}(0)$. Let us define $f : X \to [0, 1]$ by $f(x) = g(x)$ if $x \in X \setminus \operatorname{int} F$ and $f(x) = 0$ if $x \in \operatorname{int} F$. It is easy to see that f is continuous and $F = f^{-1}(0)$. Therefore, X is perfectly normal by Vedenisoff's theorem.

Remark 2. By one of reviewers, in Theorem 1 it is sufficient to assume that Y is a non-discrete metric space.

Remark 3. Any topological vector space is ω-resolvable [2]. So, the space of all continuous function $C_p([0, 1])$ equipped with the topology of pointwise convergence is an example of perfectly normal ω-resolvable space which is not metrizable.

Remark 4. Eric K. van Douwen proved in [3, Theorem 5.2] that there exists a crowded countable regular space which cannot be represented as a union of two disjoint dense subsets. It is easy to see that this space is perfectly normal and not ω-resolvable.

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