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GENERALIZED SPACES OF TYPE S AND EVOLUTIONARY PSEUDODIFFERENTIAL EQUATIONS

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We study the main operations (argument shift, differentiation, etc.) in the generalized spaces of type S and some classes of analytic functions and pseudodifferential operators in spaces of this kind, as well as the properties of the Fourier transforms of generalized functions, convolutions, convolvers, and multiplicators. The correct solvability of the nonlocal time problem is proved for one class of pseudodifferential equations in generalized spaces of type S. Its solution is presented in the form of convolution of the fundamental solution with an initial function, which is an element of the space of generalized functions of the ultradistribution type.

Introduction

Pseudodifferential operators and equations with pseudodifferential operators are closely connected with important problems of analysis, modern mathematical physics, probability theory, and the theory of fractals. Note that differential operators, operators of fractional differentiation and integration, convolutions, etc. belong to the class of pseudodifferential operators.

A broad class of pseudodifferential operators can be formally represented in the form

$$A = J^{-1}_{\sigma\to x}[a(t,x;\sigma)J_{x\to\sigma}], \quad \{x,\sigma\}\subset\mathbb{R}, \quad t>0,$$

where a is the symbol of the operator A satisfying certain conditions and J and J^{-1} are, respectively, the direct and inverse Fourier (or Bessel) transforms. If the symbol a is an entire even function of the argument σ, then the evolutionary equations with operator A also contain singular differential equations, in particular, with the Bessel operator

$$B_\nu = \frac{d^2}{dx^2} + (2\nu+1)x^{-1}\frac{d}{dx}, \qquad \nu > -\frac{1}{2}.$$

This operator contains the expression $1/x$ and can be formally represented in the form $B_\nu = F^{-1}_{B_\nu}[-\sigma^2 F_{B_\nu}]$, where F_{B_ν} is the integral Bessel transform. If $a(t,x;\sigma)\equiv P(t,x;\sigma)$, where P is a polynomial of the variable σ for fixed t and x satisfying the "parabolicity" condition, then these equations belong to the class of parabolic equations provided that $J_{x\to\sigma} = F$ is the Fourier transform or to the class of B-parabolic equations provided that $J_{x\to\sigma} = F_{B_\nu}$. The B-parabolic equations degenerate in the limit and their inner properties are close to the properties of uniformly parabolic equations [1].

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Numerous mathematicians (Nagase, Shinkai, Tsutsumi, Shubin, Taylor, Hörmander, Kochubei, Dubins'kyi, Ptashnyk, and others) studied the Cauchy problem for evolutionary equations with pseudodifferential operators by using various methods and approaches and obtained important results concerning the solvability of the Cauchy problem in different function spaces. Moreover, the initial functions often have singularities at one or several points and admit regularization in certain spaces of generalized functions of the type of Sobolev–Schwarz distributions, ultradistributions, hyperfunctions, etc. Thus, the Cauchy problem for the indicated equations admits a natural statement also in the classes of generalized functions of finite and infinite orders.

As a generalization of the Cauchy problem, we can consider the following nonlocal time multipoint problem with the initial condition

$$\sum_{k=0}^{m} \alpha_k u(t,\cdot)|_{t=t_k} = f,$$

instead of $u(t,\cdot)|_{t=0} = f$, where $t_0 = 0$, $\{t_1,\ldots,t_m\} \subset (0,T]$, $0 < t_1 < t_2 < \ldots < t_m \le T$, $\{\alpha_0,\alpha_1,\ldots,\alpha_m\} \subset \mathbb{R}$, $m \in \mathbb{N}$, are fixed numbers (for $\alpha_0 = 1$ and $\alpha_1 = \ldots = \alpha_m = 0$, we get the Cauchy problem); the corresponding condition is interpreted either in the classical sense or in the weak sense if f is a generalized function, i.e., as the limit relation

$$\sum_{k=0}^{m} \alpha_k \lim_{t\to t_k} \langle u(t,\cdot),\varphi\rangle = \langle f,\varphi\rangle$$

for an arbitrary function φ from the pivot space (here, $\langle f,\varphi\rangle$ denotes the action of the functional f upon the test function φ).

A nonlocal (in time) multipoint problem belongs to the class of nonlocal problems for differential-operator equations and partial differential equations. These problems are encountered in the simulation of various processes and practical situations by boundary-value problems for partial differential equations with nonlocal conditions, in the description of the well-defined problems for specific operators, and in the construction of the general theory of boundary-value problems (see, e.g., [2–8]).

In the present paper, we study a nonlocal multipoint (in time) problem for the equation

$$\frac{\partial u}{\partial t} + \varphi\left(\frac{i\partial}{\partial x}\right) u = 0, \quad (t,x) \in (0,\infty)\times\mathbb{R}, \tag{1}$$

where $\varphi\left(\frac{i\partial}{\partial x}\right)$ can be regarded as a pseudodifferential operator in generalized spaces of the type S constructed by using the symbol function φ satisfying certain conditions. This equation contains, e.g., the operator of differentiation of fractional order

$$\varphi\left(\frac{i\partial}{\partial x}\right) = \sqrt{I - \left(\frac{\partial}{\partial x}\right)^2}$$

constructed on the basis of the symbol function $\varphi(\sigma) = (1+\sigma^2)^{1/2}$, $\sigma \in \mathbb{R}$. We establish the correct solvability of the nonlocal multipoint (in time) problem for Eq. (1), present its solution in the form of convolution of the fundamental solution with the initial function, which is an element of the space of generalized functions of ultradistribution type, and study the properties of the fundamental solution of nonlocal problem.

Note that the spaces of type S were introduced by Gelfand and Shilov in [9]. They are used in the investigation of the problems on the classes of uniqueness and well-posedness of the Cauchy problem for partial differential equations with constant coefficients or coefficients depending solely on the time variable. Spaces of the type S

(the spaces $S_\alpha^\beta \equiv S_{k^{k\alpha}}^{n^{n\beta}}$) are constructed by using two sequences $\{k^{k\alpha}\}$, $\{n^{n\beta}\}$, $\{k,n\} \subset \mathbb{Z}_+$, where $\alpha, \beta > 0$ are fixed parameters. Elements of these spaces are functions infinitely differentiable on $\mathbb{R}$ and satisfying the condition

$$\left|x^k \varphi^{(n)}(x)\right| \le c A^k B^n k^{k\alpha} n^{n\beta}, \quad \{k,n\} \subset \mathbb{Z}_+,$$

with constants $c,\ A,\ B > 0$ that depend on the function φ. In [9], it was shown that every function from the space S_α^β, together with all its derivatives, decreases as $|x| \to +\infty$ faster than $\exp\{-a|x|^{1/\alpha}\}$, $a > 0$, $x \in \mathbb{R}$. In [10–16], it was established that the spaces of type S and S' topologically dual to S are natural sets of initial data of the Cauchy problem for broad classes of equations with partial derivatives of finite and infinite orders for which the solutions are entire functions of the space variable. It is of interest to study the spaces $S_{a_k}^{b_n}$ generalizing the spaces of type S and constructed according to the sequences $\{a_k\}$ and $\{b_n\}$ of positive numbers (i.e., to analyze the topological structures, properties of functions, and main operations in the indicated spaces). In the first part of the present paper (Secs. 1–4), we give answers to the posed problems. We also study some classes of analytic functions in the generalized spaces of type S, some classes of pseudodifferential operators in these spaces, and the properties of the Fourier transforms of generalized functions from the space S'.

1. Preliminary Information. Generalized Spaces of Type *S*

Consider a sequence of positive numbers $\{m_n, n \in \mathbb{Z}_+\}$ with the following properties:

(i) $\forall n \in \mathbb{Z}_+ \colon m_n \le m_{n+1},\ m_0 = 1$;

(ii) $\exists\, M > 0\ \exists\, h > 0\ \forall n \in \mathbb{Z}_+ \colon m_{n+1} \le M h^n m_n$;

(iii) $\exists\, \omega \ge 1\ \exists\, L \ge 1 \colon m_k m_{n-k} \le \omega L^n m_n,\ 0 \le k \le n,\ n \in \mathbb{Z}_+$.

As examples of these sequences, we can mention Gevrey sequences of the form $m_n = n^{n\alpha}$, $m_n = (n!)^\alpha$, $n \in \mathbb{Z}_+$, where $\alpha > 0$ is a fixed parameter ($0^0 = 1$).

We set

$$\gamma(x) = \inf_{n \in \mathbb{Z}_+} \frac{m_n}{|x|^n}, \qquad x \neq 0.$$

It is clear that γ is a nonnegative even function on $\mathbb{R} \setminus \{0\}$. If $x \in [-1,1] \setminus \{0\}$, then, by using property (i) of the sequence $\{m_n, n \in \mathbb{Z}_+\}$, we obtain

$$\inf_{n \in \mathbb{Z}_+} \frac{m_n}{|x|^n} = 1,$$

i.e., $\gamma(x) = 1$ for $x \in [-1,1] \setminus \{0\}$.

If $1 \le x_1 < x_2$, then $\gamma(x_2) \le \gamma(x_1) \le \gamma(1) = 1$, i.e., γ monotonically decreases in the interval $[1,+\infty)$. In view of the fact that the function γ is even on $\mathbb{R} \setminus \{0\}$, we conclude that γ monotonically increases in the interval $(-\infty,-1]$, $0 < \gamma(x) \le 1\ \forall x \in \mathbb{R} \setminus \{0\}$.

Thus, if $m_n = n^{n\alpha}$, $n \in \mathbb{Z}_+$, $\alpha > 0$, then the following estimate holds for the function γ in the interval $[1,+\infty)$ (see [9, p. 205]):

$$\gamma(\xi) = \inf_{n \in \mathbb{Z}_+} \frac{n^{n\alpha}}{\xi^n} \le e^{\frac{\alpha e}{2}} e^{-\frac{\alpha}{e}\xi^{1/\alpha}}, \quad \xi \ge 1.$$

If $0 < \xi < 1$, then

$$\inf_{n\in\mathbb{Z}_+} \frac{n^{n\alpha}}{\xi^n} = 1 \le e^{\frac{\alpha}{e}} e^{-\frac{\alpha}{e}\xi^{1/\alpha}}.$$

Thus,

$$\forall \xi\colon\ 0 < \xi < \infty\colon\ \gamma(\xi) \le c e^{-\frac{\alpha}{e}\xi^{1/\alpha}}, \quad c = e^{\frac{\alpha e}{2}}.$$

In addition, on $\mathbb{R} \setminus \{0\}$, the function γ satisfies the inequality [9, p. 204]

$$e^{-\frac{\alpha}{e}|\xi|^{1/\alpha}} \le \inf_{n\in\mathbb{Z}_+} \frac{n^{n\alpha}}{|\xi|^n} \le c e^{-\frac{\alpha}{e}|\xi|^{1/\alpha}}, \quad c = e^{\frac{\alpha e}{2}}, \quad \xi \in \mathbb{R}\setminus\{0\}. \tag{$*$}$$

Lemma 1. *The inequality*

$$\ln\gamma(x_1) + \ln\gamma(x_2) \ge \ln\gamma(x_1+x_2) \quad \forall\{x_1,x_2\} \subset (0,+\infty) \tag{2}$$

is true.

Proof. First, we note that

$$\{\gamma(x_1), \gamma(x_2), \gamma(x_1+x_2)\} \subset (0,1]$$

for any fixed $\{x_1,x_2\} \subset (0,+\infty)$. Since $\gamma(x) = 1$ for $x \in (0,1]$, it suffices to prove inequality (2) in the interval $(1,+\infty)$. Indeed, if $\{x_1,x_2\} \subset (0,1]$ and $(x_1+x_2) \in (0,1]$, then inequality (2) turns into the equality. If $\{x_1,x_2\} \subset (0,1]$ and $x_1+x_2 > 1$, then inequality (2) is also true because $0 < \gamma(x_1+x_2) < 1$ and $\ln\gamma(x_1+x_2) < 0$ and, moreover, $\gamma(x_1) = \gamma(x_2) = 1$ and $\ln\gamma(x_1) = \ln\gamma(x_2) = 0$. If $x_1 \in (0,1]$ and $x_2 > 1$, then $x_1+x_2 > 1$, $\ln\gamma(x_1) = 0$, and

$$\ln\gamma(x_1) + \ln\gamma(x_2) = \ln\gamma(x_2) \ge \ln\gamma(x_1+x_2)$$

because $\gamma(x_1+x_2) \le \gamma(x_2)$ [here, we have taken into account the fact that γ monotonically decreases in the interval $(1,+\infty)$]. Similarly, we can analyze the case $x_2 \in (0,1]$, $x_1 > 1$.

Further, let $\{x_1,x_2\} \subset (1,+\infty)$. Inequality (2) is equivalent to the inequality

$$\gamma(x_1)\gamma(x_2) \ge \gamma(x_1+x_2), \quad \{x_1,x_2\} \subset (1,+\infty). \tag{3}$$

To prove (3), it suffices to show that

$$\frac{\gamma(x_1)\gamma(x_2)}{\gamma(x_1+x_2)} \ge 1, \quad \{x_1,x_2\} \subset (1,+\infty).$$

Let $1 < x_1 \le x_2$. Since γ monotonically decreases on $(1,+\infty)$, we get $\gamma(x_1) \ge \gamma(x_2)$. Therefore,

$$\frac{\gamma(x_1)\gamma(x_2)}{\gamma(x_1+x_2)} \ge \frac{\gamma^2(x_2)}{\gamma(x_1+x_2)}.$$

By definition,

$$\gamma(x_2) = \inf_{n\in\mathbb{Z}_+} \frac{m_n}{x_2^n}, \qquad x_2 \in (1, +\infty).$$

Consider a sequence $\{\varepsilon_k = \beta_k \gamma(x_2),\ k \in \mathbb{N}\}$, where $\{\beta_k, k \in \mathbb{N}\}$ is a sequence of positive numbers monotonically approaching zero. Hence, for $\varepsilon_k > 0$, there exists a number $n_k = n_k(\varepsilon_k)$ such that

$$\frac{m_{n_k}}{x_2^{n_k}} < \gamma(x_2) + \varepsilon_k = (1 + \beta_k)\gamma(x_2),$$

i.e.,

$$\gamma(x_2) > \frac{1}{1+\beta_k} \frac{m_{n_k}}{x_2^{n_k}}, \quad k \in \mathbb{N}.$$

Thus,

$$\gamma(x_1 + x_2) \le \frac{m_{n_k}}{(x_1 + x_2)^{n_k}}, \quad k \in \mathbb{N}.$$

By using these inequalities, we conclude that, for the sequence of numbers $\{n_k, k \in \mathbb{N}\}$, the following inequality is true:

$$\frac{\gamma(x_1)\gamma(x_2)}{\gamma(x_1+x_2)} \ge \frac{\gamma^2(x_2)}{\gamma(x_1+x_2)} \ge \frac{m_{n_k}^2}{(1+\beta_k)^2} \frac{(x_1+x_2)^{n_k}}{x_2^{2n_k} m_{n_k}} \ge \frac{m_{n_k}}{(1+\beta_1)^2 x_2^{n_k}}, \quad k \in \mathbb{N}$$

(here, we have taken into account the fact that $x_1 + x_2 \ge x_2$ and $\beta_k < \beta_1\ \forall k \ge 2$). In addition,

$$\gamma(\alpha) \le \frac{m_n}{\alpha^n}, \quad n \in \mathbb{Z}_+,$$

for any $\alpha > 1$ or

$$\forall \alpha > 1 \quad \forall k \in \mathbb{N}\colon\ m_{n_k} \ge \alpha^{n_k}\gamma(\alpha).$$

We set $\alpha = x_2\delta$, where $\delta > 1$ is a fixed number, and choose a number n_k such that the inequality

$$\delta^{n_k}\gamma(x_2\delta) \ge (1+\beta_1)^2$$

is true. We directly obtain

$$n_k \ge \left[\ln\left(\frac{(1+\beta_1)^2}{\gamma(x_2\delta)}\right)(\ln\delta)^{-1} + 1\right].$$

For this number, the inequality

$$\frac{\gamma(x_1)\gamma(x_2)}{\gamma(x_1+x_2)} \ge \frac{x_2^{n_k}\delta^{n_k}\gamma(x_2\delta)}{(1+\beta_1)^2 x_2^{n_k}} = \frac{\delta^{n_k}\gamma(x_2\delta)}{(1+\beta_1)^2} \ge 1$$

is true, Q.E.D.

Let $\{a_k, k \in \mathbb{Z}_+\}$ and $\{b_n, n \in \mathbb{Z}_+\}$ be sequences with properties (i)–(iii). By $S_{a_k}^{b_n}$ we denote a collection of functions $\varphi \in C^\infty(\mathbb{R})$ satisfying the condition

$$\exists\, c, A, B > 0 \quad \forall \{k,n\} \subset \mathbb{Z}_+ \quad \forall x \in \mathbb{R}\colon \; \left|x^k \varphi^{(n)}(x)\right| \le c A^k B^n a_k b_n \tag{4}$$

(the constants $c,\ A,\ B > 0$ depend on the function φ).

The collection $S_{a_k}^{b_n}$ coincides with the union of the spaces $S_{a_k,A}^{b_n,B}$ over all indices $\{A, B\} \subset \mathbb{N}$, where $S_{a_k,A}^{b_n,B}$ denotes the collection of functions $\varphi \in S_{a_k}^{b_n}$ satisfying, for any $\delta, \rho > 0$, the inequalities

$$\left|x^k \varphi^{(n)}(x)\right| \le c_{\delta\rho} (A+\delta)^k (B+\rho)^n a_k b_n, \quad \{k,n\} \subset \mathbb{Z}_+, \quad x \in \mathbb{R},$$

with the same constants $A, B > 0$. Hence, $S_{a_k,A}^{b_n,B}$ turns into a complete countably normed space if a system of norms in this space is given by the formulas

$$\|\varphi\|_{\delta\rho} = \sup_{x,k,n} \frac{\left|x^k \varphi^{(n)}(x)\right|}{(A+\delta)^k (B+\rho)^n a_k b_n}, \quad \{\delta, \rho\} \subset \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}.$$

The sequence $\{\varphi_\nu, \nu \ge 1\} \subset S_{a_k}^{b_n}$ converges to zero in this space if the functions φ_ν and their derivatives of any order uniformly converge to zero on each segment $[a, b] \subset \mathbb{R}$ and the following inequalities are true:

$$\left|x^k \varphi_\nu^{(n)}(x)\right| \le c A^k B^n a_k b_n, \quad \{k,n\} \subset \mathbb{Z}_+, \quad x \in \mathbb{R},$$

where the constants $c,\ A,\ B > 0$ are independent of ν.

Lemma 2. *A function $\varphi \in C^\infty(\mathbb{R})$ is an element of the space $S_{a_k}^{b_n}$ if and only if it satisfies the condition*

$$\exists\, c, a, B > 0 \quad \forall x \in \mathbb{R} \setminus \{0\} \quad \forall \{k,n\} \subset \mathbb{Z}_+\colon$$

$$\left|\varphi^{(n)}(x)\right| \le c B^n b_n \gamma(ax), \quad \left|\varphi^{(n)}(0)\right| \le C B^n b_n, \tag{5}$$

where

$$\gamma(x) = \inf_{k \in \mathbb{Z}_+} \frac{a_k}{|x|^k}, \qquad x \in \mathbb{R} \setminus \{0\}.$$

Proof. Let $\varphi \in S_{a_k}^{b_n}$, i.e., condition (4) is satisfied. Dividing both sides of inequality (4) by $|x|^k$, $x \ne 0$, and passing to the lower limit with respect to k on the right-hand side, we obtain

$$\left|\varphi^{(n)}(x)\right| \le c B^n b_n \inf_{k \in \mathbb{Z}_+} \frac{A^k a_k}{|x|^k} = c B^n b_n \inf_{k \in \mathbb{Z}_+} \frac{a_k}{\left|A^{-1} x\right|^k} = c B^n b_n \gamma(ax), \quad x \ne 0,$$

where $a = A^{-1} > 0$. Moreover, (4) yields the estimate

$$\left|\varphi^{(n)}(0)\right| \le c B^n b_n, \qquad n \in \mathbb{Z}_+.$$

Conversely, let the function $\varphi \in C^\infty(\mathbb{R})$ satisfy condition (5). Then

$$\left|\varphi^{(n)}(x)\right| \le cB^n b_n \inf_{k\in\mathbb{Z}_+} \frac{a_k}{|ax|^k}, \quad n \in \mathbb{Z}_+, \quad x \ne 0.$$

This yields the inequality

$$\forall x \in \mathbb{R} \setminus \{0\}\colon\ |ax|^k \left|\varphi^{(n)}(x)\right| \le cB^n b_n a_k, \quad \{k, n\} \subset \mathbb{Z}_+.$$

By using the estimate $\left|\varphi^{(n)}(0)\right| \le cB^n b_n$, $n \in \mathbb{Z}_+$, we get

$$\left|x^k \varphi^{(n)}(x)\right| \le cA^k B^n a_k b_n, \quad A = a^{-1} \quad \forall \{k, n\} \subset \mathbb{Z}_+ \quad \forall x \in \mathbb{R},$$

Q.E.D.

For a function

$$\gamma_1(x) = \begin{cases} 1, & |x| \le 1, \\ \inf\limits_{k\in\mathbb{Z}_+} \dfrac{a_k}{|x|^k}, & x > 1, \end{cases}$$

condition (5) can be replaced by the equivalent condition

$$\exists\, c, a, B > 0 \quad \forall x \in \mathbb{R} \quad \forall n \in \mathbb{Z}_+\colon\ \left|\varphi^{(n)}(x)\right| \le cB^n b_n \gamma_1(ax).$$

If

$$a_k = k^{k\alpha}, \quad b_n = n^{n\beta}, \quad \{k, n\} \subset \mathbb{Z}_+,$$

where α, $\beta > 0$ are fixed parameters, then we denote the space $S_{k^{k\alpha}}^{n^{n\beta}}$ by S_α^β. The spaces S_α^β are called spaces of the type S. These spaces were studied in detail in the monograph [9]. They can be also characterized as in [9, p. 210]:

The space S_α^β consists of those and only those functions that are infinitely differentiable on $\mathbb{R}$ and satisfy the inequalities

$$\left|\varphi^{(n)}(x)\right| \le cB^n n^{n\beta} \exp\left\{-a|x|^{1/\alpha}\right\}, \quad n \in \mathbb{Z}_+, \quad x \in \mathbb{R},$$

with some constants c, a, $B > 0$ that depend only on the function φ.

If $0 < \beta < 1$ and $\alpha \ge 1 - \beta$, then S_α^β consists of those and only those functions $\varphi \in C^\infty(\mathbb{R})$ that can be analytically extended to the entire complex plane and satisfy the inequality

$$|\varphi(x + iy)| \le c \exp\left\{-a|x|^{1/\alpha} + b|y|^{1/(1-\beta)}\right\}, \quad c, a, b > 0.$$

The space S_α^1 consists of functions $\varphi \in C^\infty(\mathbb{R})$ that can be analytically extended into a certain strip $|\operatorname{Im} z| < \delta$ (that depends on φ) of the complex plane and, in addition, the estimate

$$|\varphi(x + iy)| \le c \exp\left\{-a|x|^{1/\alpha}\right\}, \quad c, a > 0, \quad \{x, y\} \subset \mathbb{R},$$

is true.

A set $F \subset S_{a_k}^{b_n}$ is called bounded in this space if $F \subset S_{a_k,A}^{b_n,B}$ with some constants $A, B > 0$, i.e., for all functions $\varphi \in F$, estimates (4) are true with the same constants $A, B > 0$.

The spaces $S_{a_k}^{b_n}$ are called generalized spaces of type S. In the generalized spaces of type S, linear and continuous operators of shift of the argument, multiplication by an independent variable, and differentiation are defined.

Thus, we can show that, in the space $S_{a_k}^{b_n}$, the operator of shift of the argument $\varphi(x) \to \varphi(x-h)$ $\forall \varphi \in S_{a_k}^{b_n}$ that maps this space into itself is defined and continuous.

Assume that φ runs through the bounded set $F \subset S_{a_k}^{b_n}$. This means that, for each function $\varphi \in F$, the inequalities

$$\left|x^k \varphi^{(n)}(x)\right| \le cA^k B^n a_k b_n, \quad \{k,n\} \subset \mathbb{Z}_+, \quad x \in \mathbb{R},$$

are true with some constants $c, A, B > 0$. Thus, we get

$$\begin{aligned}
\sup_{x\in\mathbb{R}} \left|x^k \varphi^{(n)}(x-h)\right| &= \sup_{x\in\mathbb{R}} |(x+h)^k \varphi^{(n)}(x)| = \sup_{x\in\mathbb{R}} \left|\sum_{j=0}^{k} C_k^j x^j h^{k-j} \varphi^{(n)}(x)\right| \\
&\le \sum_{j=0}^{k} C_k^j |h|^{k-j} \sup_{x\in\mathbb{R}} \left|x^j \varphi^{(n)}(x)\right| \\
&\le c \sum_{j=0}^{k} C_k^j |h|^{k-j} A^j B^n a_j b_n = cB^n b_n \sum_{j=0}^{k} C_k^j A^j |h|^{k-j} \\
&\le cB^n a_k b_n \sum_{j=0}^{k} C_k^j A^j |h|^{k-j} = c(A+|h|)^k B^n a_k b_n = cA_1^k B^n a_k b_n,
\end{aligned}$$

where $A_1 = A + |h|$. This implies that the function $\varphi_1(x) = \varphi(x-h)$ is an element of the space $S_{a_k,A+|h|}^{b_n,B}$, i.e.,

$$\varphi_1 \in S_{a_k}^{b_n} = \bigcup_{A,B>0} S_{a_k,A}^{b_n,B}.$$

Thus, the image of a bounded set F under the indicated mapping is a bounded set from the space $S_{a_k}^{b_n}$. This means that the operator of shift of the argument is a linear bounded operator in the space $S_{a_k}^{b_n}$ and, hence, also a linear continuous operator in this space because the first axiom of countability is true in the space $S_{a_k}^{b_n}$. As follows from the general theory of linear continuous operators in countably normed spaces (see [9, pp. 81–82]), the class of linear bounded operators in the spaces with the first axiom of countability coincides with the class of linear continuous operators.

We also note that the spaces $S_{a_k}^{b_n}$ are perfect (i.e., spaces all bounded sets in which are compact). In view of this fact and the general theory of perfect spaces (see [9, p. 171]), the operation of shift of the argument is differentiable (and even infinitely differentiable) in a sense that the limit relations of the form

$$(\varphi(x+h) - \varphi(x))h^{-1} \to \varphi'(x), \qquad h \to 0,$$

are true for each function $\varphi \in S_{a_k}^{b_n}$ in a sense of convergence in the topology of the space $S_{a_k}^{b_n}$.

We now prove that a linear and continuous operator of multiplication by an integer independent variable that maps this space into itself is defined in the space $S_{a_k}^{b_n}$.

Assume that φ runs through a bounded set $F \subset S_{a_k}^{b_n}$, i.e., each function $\varphi \in F$ satisfies the inequality

$$\left|x^k \varphi^{(n)}(x)\right| \le c A^k B^n a_k b_n, \quad \{k, n\} \subset \mathbb{Z}_+, \quad x \in \mathbb{R},$$

with some constants $c, A, B > 0$. We set $\psi(x) := x\varphi(x)$. Then

$$\begin{aligned}\left|x^k \psi^{(n)}(x)\right| &= \left|x^k (x\varphi(x))^{(n)}\right| \\ &\le \left|x^{k+1}\varphi^{(n)}(x)\right| + n\left|x^k \varphi^{(n-1)}(x)\right| \\ &\le cA^{k+1}B^n a_{k+1} b_n + ncA^k B^{n-1} a_k b_{n-1}.\end{aligned}$$

By using property (ii) of the sequence $\{a_k, k \in \mathbb{Z}_+\}$ and property (i) of the sequence $\{b_n, n \in \mathbb{Z}_+\}$, we arrive at the inequalities

$$\left|x^k \psi^{(n)}(x)\right| \le cAA^k B^n M h^k a_k b_n + c2^n A^k B^n B^{-1} a_k b_n = \tilde{c} A_1^k B_1^n a_k b_n,$$

where $\tilde{c} = cAM + cB^{-1}$, $A_1 = \max\{Ah, A\}$, and $B_1 = \max\{1, 2B\}$. Hence, a bounded set F multiplied by an independent variable x is also a bounded set in the space $S_{a_k}^{b_n}$, Q.E.D.

We also note that $\varphi\psi \in S_{a_k}^{b_n}$ for any $\{\varphi, \psi\} \subset S_{a_k}^{b_n}$.

A function $g \in C^\infty(\mathbb{R})$ is called a multiplicator in the space $S_{a_k}^{b_n}$ if $g\varphi \in S_{a_k}^{b_n}$ for any function $\varphi \in S_{a_k}^{b_n}$ and $\varphi \to g\varphi$ is a linear and continuous mapping that acts from $S_{a_k}^{b_n}$ into $S_{a_k}^{b_n}$.

Lemma 3. *A multiplicator in the space $S_{a_k}^{b_n}$ is a function $f \in C^\infty(\mathbb{R})$ satisfying the condition*

$$\exists\, B_0 > 0 \quad \forall \varepsilon > 0 \quad \exists c_\varepsilon > 0 \quad \forall n \in \mathbb{Z}_+ \quad \forall x \in \mathbb{R} \setminus \{0\}:$$

$$\left|f^{(n)}(x)\right| \le c_\varepsilon B_0^n b_n (\gamma(\varepsilon x))^{-1}, \tag{6}$$

$$\left|f^{(n)}(0)\right| \le c_\varepsilon B_0^n b_n.$$

Proof. Let $\varphi \in S_{a_k}^{b_n}$. By Lemma 2, the inequalities

$$\left|\varphi^{(n)}(x)\right| \le cB^n b_n \gamma(ax), \quad x \in \mathbb{R} \setminus \{0\}, \quad n \in \mathbb{Z}_+, \quad \left|\varphi^{(n)}(0)\right| \le cB^n b_n$$

are true with some constants $c, A, B > 0$. We take $\varepsilon \in (0, a)$ and use estimates (6). Thus, we get

$$\left|(f(x)\varphi(x))^{(n)}\right| \le \sum_{j=0}^{n} C_n^j \left|f^{(j)}(x)\right| \left|\varphi^{(n-j)}(x)\right|$$

$$\leq cc_\varepsilon \sum_{j=0}^{n} C_n^j B_0^j B^{n-j} b_{n-j} \frac{\gamma(ax)}{\gamma(\varepsilon x)}, \quad x \neq 0.$$

Since $b_j b_{n-j} \leq \omega L^n b_n$ [see property (iii) of the sequence $\{b_n, n \in \mathbb{Z}_+\}$], we get

$$\left|(f(x)\varphi(x))^{(n)}\right| \leq \tilde{c} B_1^n b_n \frac{\gamma(ax)}{\gamma(\varepsilon x)} = \tilde{c} B_1^n b_n e^{\ln \gamma(ax) - \ln \gamma(\varepsilon x)}, \quad x \neq 0,$$

where $\tilde{c} = cc_\varepsilon \omega$ and $B_1 = 2\max\{B_0, B\}L$. The inequality

$$\ln \gamma(ax) - \ln \gamma(\varepsilon x) \leq \ln \gamma((a-\varepsilon)x), \quad 0 < \varepsilon < a, \quad x \neq 0,$$

follows from (2). Thus, we obtain

$$\left|(f(x)\varphi(x))^{(n)}\right| \leq \tilde{c} B_1^n b_n e^{\ln \gamma((a-\varepsilon)x)} = \tilde{c} B_1^n b_n \gamma(a_1 x), \quad a_1 = a - \varepsilon, \quad x \neq 0,$$

$$\left|(f(x)\varphi(x))^{(n)}|_{x=0}\right| \leq \tilde{c} B_1^n b_n, \quad n \in \mathbb{Z}_+.$$

Hence, $f \cdot \varphi \in S_{a_k}^{b_n}$. It follows from the presented reasoning that if φ runs through a bounded set $F \subset S_{a_k}^{b_n}$, then every function $f \cdot \varphi$, $\varphi \in F$ belongs to a bounded set $F_1 \subset S_{a_k}^{b_n}$, i.e., the operator $S_{a_k}^{b_n} \ni \varphi \to f\varphi \in S_{a_k}^{b_n}$ is continuous.

Lemma 3 is proved.

If the sequences $\{a_k, k \in \mathbb{Z}_+\}$ and $\{b_n, n \in \mathbb{Z}_+\}$ satisfy the condition

$$\frac{a_k}{a_{k-1}} \geq c_a k^{1-\mu}, \quad \frac{b_k}{b_{k-1}} \geq c_b k^{1-\lambda}, \quad \mu, \lambda \geq 0, \quad \mu + \lambda \leq 1,$$
$$\{k, n\} \subset \mathbb{N}, \quad \frac{a_{k+2}}{a_k} \leq c_0 A^k, \tag{7}$$

then, as indicated in [9, p. 290], the following formula is true:

$$F\left[S_{a_k}^{b_n}\right] = S_{b_k}^{a_n},$$

where

$$F\left[S_{a_k}^{b_n}\right] := \left\{\psi \colon \psi(\sigma) = \int_{\mathbb{R}} \varphi(\sigma) e^{i\sigma x}\, dx \ \ \forall \varphi \in S_{a_k}^{b_n}\right\}.$$

In particular,

$$F\left[S_{k^{k\alpha}}^{n^{n\beta}}\right] \equiv F\left[S_\alpha^\beta\right] = S_\beta^\alpha = S_{k^{k\beta}}^{n^{n\alpha}}, \quad \alpha, \beta > 0.$$

The operator $F\colon S_{a_k}^{b_n} \to S_{b_k}^{a_n}$ is continuous.

Since the sequence $\{a_k, k \in \mathbb{Z}_+\}$ has property (ii), the inequality

$$\frac{a_{k+2}}{a_k} \le c_0 A_0^k$$

is true with constants $c_0 = hM^2$ and $A_0 = h^2$ [$M, h > 0$ are constants from property (ii)]. In what follows, we assume that the sequences $\{a_k, k \in \mathbb{Z}_+\}$ and $\{b_n, n \in \mathbb{Z}_+\}$ satisfy condition (7).

If $a_k = k^{k\alpha}$, $k \in \mathbb{Z}_+$, $\alpha > 0$, then, as shown in [9, p. 240], this sequence satisfies the inequality

$$\frac{a_k}{a_{k-1}} \ge \frac{1}{2^\alpha} k^{1-\mu}, \qquad k \in \mathbb{N},$$

where $\mu = \max\{1-\alpha, 0\}$, i.e., $\mu = 1-\alpha$ for $0 < \alpha < 1$ and $\mu = 0$ for $\alpha \ge 1$.

2. Some Classes of Analytic Functions in the Generalized Spaces of Type S

If a sequence $\{b_n, n \in \mathbb{Z}_+\}$ used to construct the space $S_{a_k}^{b_n}$ has a special form, then this space may consist of infinitely differentiable functions on $\mathbb{R}$ admitting analytic extensions to the entire complex plane and characterized by a certain order of increase. Assume that a sequence $\{a_k, k \in \mathbb{Z}_+\}$ satisfies conditions (i)–(iii) and that a sequence $\{b_n, n \in \mathbb{Z}_+\}$ is characterized by "slow growth." In order to construct the sequence $\{b_n\}$, we consider a monotonically decreasing sequence $\{\rho_n, n \in \mathbb{Z}_+\}$, $\rho_0 = 1$, of positive numbers with the following properties:

(a) $\lim\limits_{n\to\infty} \sqrt[n]{\rho_n} = 0$;

(b) $\exists\, \gamma \in (0,1)\ \forall n \in \mathbb{N}\colon \rho_{n-1}/\rho_n \le n^\gamma$;

(c) $\exists\, c, L \ge 1\ \rho_k \rho_{n-k} \le c L^n \rho_n$, $k \in \{0, 1, \ldots, n\}$.

As an example of a sequence $\{\rho_n, n \in \mathbb{Z}_+\}$ with properties (a)–(c), we can mention a sequence $\rho_n = (n!)^{\beta-1}$, where $\beta \in (0,1)$ is a fixed parameter. Indeed, first of all, we note that $\{\rho_n\}$ is a monotonically decreasing sequence, i.e.,

$$\rho_n = \frac{1}{(n!)^{1-\beta}} \ge \frac{1}{((n+1)!)^{1-\beta}} = \rho_{n+1} \quad \forall n \in \mathbb{Z}_+.$$

By using the Stirling formula, we obtain

$$\sqrt[n]{\rho_n} = \frac{1}{(n!)^{(1-\beta)/n}} = \frac{1}{\left(\sqrt{2\pi n}\, n^n e^{-n} e^{-n} e^{\theta/12n}\right)^{(1-\beta)/n}}$$

$$\le \frac{1}{(n^n e^{-n})^{(1-\beta)/n}} = \frac{e^{1-\beta}}{n^{1-\beta}} \to 0, \quad n \to \infty, \quad 0 < \theta < 1.$$

Thus, the sequence $\{\rho_n, n \in \mathbb{Z}_+\}$ has property (a).

For any $n \in \mathbb{N}$, we get

$$\frac{\rho_{n-1}}{\rho_n} = \frac{(n!)^{1-\beta}}{((n-1)!)^{1-\beta}} = n^{1-\beta}.$$

Hence, the sequence $\{\rho_n, n \in \mathbb{Z}_+\}$ satisfies condition (b) with the parameter $\gamma = 1-\beta$, $\gamma \in (0,1)$.

To prove property (c), it suffices to establish the inequality

$$\frac{\rho_k \rho_{n-k}}{\rho_n} \le c L^n, \quad k \in \{0,1,\dots,n\},$$

where $c, L \ge 1$, are constants. Thus,

$$\frac{\rho_k \rho_{n-k}}{\rho_n} = \frac{(n!)^{1-\beta}}{((k)!)^{1-\beta}((n-k)!)^{1-\beta}} = \left(C_n^k\right)^{1-\beta},$$

where C_n^k are the binomial coefficients in the binomial formula. Further, by using the fact that

$$\sum_{k=0}^{n} C_n^k = 2^n,$$

we obtain

$$C_n^k \le 2^n, \quad k \in \{0,1,\dots,n\}, \quad \left(C_n^k\right)^{1-\beta} \le 2^{n(1-\beta)}.$$

This yields the inequality

$$\rho_k \rho_{n-k} \le c L^n \rho_n, \quad c = 1, \qquad L = 2^{1-\beta} \ge 1, \quad k \in \{0,1,\dots,n\}.$$

We now consider a sequence $b_n = n!\rho_n$, $n \in \mathbb{Z}_+$, where $\{\rho_n\}$ is a sequence with properties (a)–(c). The sequence $\{b_n, n \in \mathbb{Z}_+\}$ has properties (i)–(iii) (see Sec. 1). Indeed, the inequality $b_n \le b_{n+1}$, $n \in \mathbb{Z}_+$, is equivalent to the inequality $b_n/b_{n+1} \le 1$, $n \in \mathbb{Z}_+$. By using property (b) of the sequence $\{\rho_n\}$,we get

$$\frac{b_n}{b_{n+1}} = \frac{n!\rho_n}{(n+1)!\rho_{n+1}} = \frac{1}{n+1}\frac{\rho_n}{\rho_{n+1}} \le \frac{(n+1)^\gamma}{n+1} = \frac{1}{(n+1)^{1-\gamma}} \le 1 \quad \forall n \in \mathbb{Z}_+.$$

Since the sequence $\{\rho_n\}$ is monotonically decreasing, we have

$$b_{n+1} = (n+1)!\rho_{n+1} = n!(n+1)\rho_{n+1} \le n!\rho_n(n+1) = (n+1)b_n \le 2^n b_n.$$

Hence, the sequence $\{b_n\}$ satisfies condition (ii) with the parameters $M = 1$ and $h = 2$. By using the inequality $k!(n-k)! \le n!$, $k \in \{0,1,\dots,n\}$, and property (c) of the sequence $\{\rho_n\}$, we get

$$b_k b_{n-k} = k!\rho_k (n-k)!\rho_{n-k} \le c L^n n!\rho_n = c L^n b_n, \quad k \in \{0,1,\dots,n\}.$$

Thus, the sequence $\{b_n\}$ satisfies condition (iii).

Theorem 1. *Every function $\varphi \in S_{a_k}^{n!\rho_n}$ can be analytically extended onto the complex plane as an entire function $\varphi(z)$, $z = x + iy \in \mathbb{C}$, satisfying the condition*

$$\exists\, a,b,c > 0 \quad \forall z = x+iy \in \mathbb{C}\colon\ |\varphi(x+iy)| \le c\gamma_1(ax)\rho_1(by),$$

where

$$\gamma_1(x) = \begin{cases} 1, & |x| \le 1, \\ \inf\limits_{k\in\mathbb{Z}_+} \left(a_k/|x|^k\right), & |x| > 1, \end{cases} \qquad \rho_1(y) = \begin{cases} 1, & y = 0, \\ \sup\limits_{n\in\mathbb{Z}_+} \left(|y|^n \rho_n\right), & y \ne 0. \end{cases}$$

Proof. Let $\varphi \in S_{a_k}^{n!\rho_n}$, i.e.,

$$\exists\, c_1, A, B > 0 \quad \forall x \in \mathbb{R} \quad \forall \{k, n\} \subset \mathbb{Z}_+\colon\ \left|x^k \varphi^{(n)}(x)\right| \le c_1 A^k B^n b_k n! \rho_n,$$

or

$$\exists\, c_1, a, B > 0 \quad \forall x \in \mathbb{R} \quad \forall n \in \mathbb{Z}_+\colon\ \left|\varphi^{(n)}(x)\right| \le c_1 B^n n! \rho_n \gamma_1(ax).$$

Therefore, it can be analytically extended onto the entire complex plane. Indeed, the remainder of the Taylor formula

$$\varphi(x+h) = \sum_{m=0}^{n-1} \frac{\varphi^{(m)}(x)}{m!}\, h^m + \frac{\varphi^{(n)}(\xi)}{n!}\, h^n, \quad x \in \mathbb{R},$$

where $|x - \xi| < |h|$, admits the following estimate:

$$\frac{\left|\varphi^{(n)}(\xi)\right|}{n!}\, |h|^n \le c_1 B^n \rho_n |h|^n = c_1 \left(B|h| \sqrt[n]{\rho_n}\right)^n.$$

Since $\sqrt[n]{\rho_n} \to 0$ as $n \to \infty$, we have

$$\forall \varepsilon > 0 \quad \exists\, n_0 = n_0(\varepsilon) \quad \forall n \ge n_0\colon\ \rho_n < \varepsilon^n.$$

We fix arbitrary $|h| > 0$ and set $\varepsilon = \dfrac{1}{2}\,(B|h|)$. Hence, we get

$$\frac{\left|\varphi^{(n)}(\xi)\right|}{n!}\, |h|^n \le \frac{c_1}{2^n} \to 0, \quad n \to \infty.$$

This implies that the Taylor series of the function $\varphi(x)$ converges to $\varphi(x)$, $x \in \mathbb{R}$, i.e.,

$$\varphi(x+h) = \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(x)}{n!}\, h^n.$$

By using the estimate for the remainder, we conclude that the Taylor series of the function φ is also convergent for complex values of h. Thus, φ can be extended to an entire function $\varphi(z)$, $z = x + iy \in \mathbb{C}$, $y \ne 0$. Moreover,

$$x^k \varphi(x+iy) = \sum_{n=0}^{\infty} \frac{(iy)^n}{n!}\, x^k \varphi^{(n)}(x)$$

and

$$\left|x^k\varphi(x+iy)\right| \le c_1 A^k a_k \sum_{n=0}^{\infty} |y|^n B^n \rho_n$$

or

$$|\varphi(x+iy)| \le c_1 \sum_{n=0}^{\infty} \left|y^n\right| B^n \rho_n \gamma_1(ax) = c_1 \sum_{n=0}^{\infty} \frac{1}{2^n} (2B|y|)^n \rho_n \gamma_1(ax)$$

$$\le c_1 \sup_{n\in\mathbb{Z}_+} \left(|by|^n \rho_n\right) \sum_{n=0}^{\infty} \frac{1}{2^n} \gamma_1(ax) = c\gamma_1(ax)\rho_1(by),$$

where $b = 2B$, $c = 2c_1$, and $\rho_1(y) = \sup_{n\in\mathbb{Z}_+} \left(|y|^n \rho_n\right)$, $y \ne 0$. Q.E.D.

Note that ρ_1 is an even nonnegative function on $\mathbb{R}$ monotonically increasing in the interval $[1, +\infty)$, $\rho_1(y) = 1$ for $|y| \le 1$, and $\rho_1(y) \ge y^n \rho_n \ \forall n \in \mathbb{Z}_+$ for $|y| > 1$.

As an example, we consider the sequences $a_k = k^{k\alpha}$ and $b_n = n!(n!)^{\beta-1} = (n!)^{\beta}$, where $\alpha \ge 1-\beta$, $\beta \in (0,1)$. In this case,

$$S_{a_k}^{b_n} = S_{k^{k\alpha}}^{(n!)^{\beta}} = S_{k^{k\alpha}}^{n^{n\beta}} = S_{\alpha}^{\beta}$$

(the inequality $\alpha \ge 1-\beta$ is the condition of nontriviality of the space S_α^β [9, p. 276]). Thus,

$$\rho_1(y) = \sup_{n\in\mathbb{Z}_+} \left(|y|^n \rho_n\right) = \sup_{n\in\mathbb{Z}_+} \left(|y|^n (n!)^{\beta-1}\right) = \sup_{n\in\mathbb{Z}_+} \frac{|y|^n}{(n!)^{1-\beta}}$$

$$\le \sup_{n\in\mathbb{Z}_+} \frac{|ey|^n}{n^{n(1-\beta)}} = \frac{1}{\inf\limits_{n\in\mathbb{Z}_+} \dfrac{n^{n(1-\beta)}}{|ey|^n}} \le e^{b_1|y|^{1/(1-\beta)}}, \quad b_1 = \frac{1-\beta}{e} e^{1/(1-\beta)}$$

(here, we have used inequality (*) in Sec. 1). Moreover, it follows from this inequality that

$$\gamma_1(x) \le c_0 \exp\left(-c_1|x|^{1/\alpha}\right), \quad c_1 = \frac{\alpha}{e}, \quad \text{and} \quad c_0 = e^{\alpha e/2}.$$

Thus, for the space S_α^β, $\alpha \ge 1-\beta$, by virtue of Theorem 1, every function φ from this space can be analytically extended onto the complex plane as an entire function $\varphi(x+iy)$ satisfying the condition

$$|\varphi(x+iy)| \le c \exp\left\{-a|x|^{1/\alpha} + b|y|^{1/(1-\beta)}\right\}$$

with some constants $c, a, b > 0$ (this condition is the well-known result established in [9, p. 209]).

3. Spaces of Generalized Functions of the Type S'

By $\left(S_{a_k}^{b_n}\right)'$ we denote the space of all linear continuous functionals defined in the pivot space $S_{a_k}^{b_n}$ with weak convergence. The elements of this space are called generalized functions.

Linear continuous functionals whose action upon a test function $\varphi \in S_{a_k}^{b_n}$ is given by the formula

$$\langle f, \varphi \rangle = \int_{\mathbb{R}} f(x)\varphi(x)dx$$

are called regular generalized functions or regular functionals.

Every function f locally integrable on $\mathbb{R}$ and satisfying the condition

$$\forall \varepsilon > 0 \quad \exists\, c_\varepsilon > 0 \quad \forall x \in \mathbb{R} \setminus \{0\}\colon \ |f(x)| \le c_\varepsilon(\gamma(\varepsilon x))^{-1}, \tag{8}$$

generates a regular generalized function $F_f \in (S_{a_k}^{b_n})'$:

$$\langle F_f, \varphi \rangle = \int_{\mathbb{R}} f(x)\varphi(x)dx \quad \forall \varphi \in S_{a_k}^{b_n}.$$

The following statement is true: *Let f and g be functions locally integrable on $\mathbb{R}$ and satisfying condition (8). Assume that these functions do not coincide on a set of positive Lebesgue measure. Then there exists a function $\varphi_0 \in S_{a_k}^{b_n}$ such that $\langle f, \varphi_0 \rangle \neq \langle g, \varphi_0 \rangle$, i.e., $F_f \neq F_g$. Conversely, if $F_f \neq F_g$, then the functions f and g do not coincide on a set of positive Lebesgue measure.*

The proof of this statement is similar to the proof of the corresponding assertion in [17].

The formulated statement enables us to identify the functions locally integrable on $\mathbb{R}$ and satisfying condition (8) with generalized functions from the space $\left(S_{a_k}^{b_n}\right)'$ generated by these functions. By using the properties of the Lebesgue integral, we conclude that the embedding

$$S_{a_k}^{b_n} \ni f \to F_f \in \left(S_{a_k}^{b_n}\right)'$$

is continuous.

Since the operation of shift of the argument T_x is defined in the pivot space $S_{a_k}^{b_n}$, we introduce the convolution of a generalized function $f \in \left(S_{a_k}^{b_n}\right)'$ with a test function by the formula

$$(f * \varphi)(x) := \left\langle f_\xi, T_{-x}\check{\varphi}(x)\right\rangle = \langle f_\xi, \varphi(x-\xi)\rangle, \quad \check{\varphi}(\xi) = \varphi(-\xi)$$

(here, $\langle f_\xi, T_{-x}\check{\varphi}(\xi)\rangle$ denotes the action of the functional f upon the test function $T_{-x}\check{\varphi}(\xi)$ as a function of the argument ξ). It follows from the property of infinite differentiability of the operation of shift of the argument in the space $S_{a_k}^{b_n}$ that the convolution $f * \varphi$ is an ordinary infinitely differentiable function on $\mathbb{R}$.

Let $f \in \left(S_{a_k}^{b_n}\right)'$. If $f * \varphi \in S_{a_k}^{b_n}$ $\forall \varphi \in S_{a_k}^{b_n}$ and the relation $\varphi_\nu \to 0$ as $\nu \to +\infty$ in the topology of the space $S_{a_k}^{b_n}$ implies that $f * \varphi_\nu \to 0$ as $\nu \to +\infty$ in the topology of the space $S_{a_k}^{b_n}$, then the functional f is called a *convolver in the space* $S_{a_k}^{b_n}$.

We define the Fourier transform of a generalized function $f \in \left(S_{a_k}^{b_n}\right)'$ by the formula

$$\langle F[f], \varphi \rangle = \langle f, F[\varphi] \rangle \qquad \forall \varphi \in S_{b_k}^{a_n}.$$

This implies that $F[f] \in \big(S_{b_k}^{a_n}\big)'$ for $f \in \big(S_{a_k}^{b_n}\big)'$. Moreover, the operator $F\colon \big(S_{a_k}^{b_n}\big)' \to \big(S_{b_k}^{a_n}\big)'$ is continuous.

Theorem 2. *If a generalized function $f \in \big(S_{a_k}^{b_n}\big)'$ is a convolver in the space $S_{a_k}^{b_n}$, then, for any function $\varphi \in S_{a_k}^{b_n}$, the formula*

$$F[f * \varphi] = F[f]F[\varphi]$$

is true.

Proof. By the condition of the theorem, $f * \varphi \in S_{a_k}^{b_n}$ $\forall \varphi \in S_{a_k}^{b_n}$. By using the Fourier transform of a generalized function from the space $\big(S_{a_k}^{b_n}\big)'$ and the definition of convolution of a generalized function with a test function, we arrive at the relations:

$$\forall \psi \in S_{b_k}^{a_n}\colon\ \langle F[f * \varphi], \psi\rangle = \langle f * \varphi, F[\psi]\rangle = \int_{-\infty}^{+\infty} (f * \varphi)(x) F[\psi](x)dx$$

$$= \int_{-\infty}^{+\infty} \big\langle f_\xi, \varphi(x-\xi)\big\rangle\, F[\psi](x)dx$$

$$= \left\langle f_\xi, \int_{-\infty}^{+\infty} \varphi(x-\xi) F[\psi](x)dx \right\rangle. \tag{9}$$

Let

$$I(\xi) := \int_{-\infty}^{+\infty} \varphi(x-\xi) F[\psi](x)dx.$$

Then, by the Fubini theorem,

$$I(\xi) = \int_{-\infty}^{+\infty} \varphi(x-\xi) \left(\int_{-\infty}^{+\infty} \psi(\sigma) e^{i\sigma x} d\sigma \right) dx = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \varphi(x-\xi)\psi(\sigma) e^{i\sigma x} d\sigma dx$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \varphi(t)\psi(\sigma) e^{i\sigma t} e^{i\sigma\xi} d\sigma dt = \int_{-\infty}^{+\infty} \psi(\sigma) F[\varphi](\sigma) e^{i\sigma\xi} d\sigma = F[F[\varphi]\psi](\xi) \in S_{a_k}^{b_n}.$$

Hence,

$$\langle F[f * \varphi], \psi\rangle = \langle f, F[F[\varphi]\psi]\rangle = \langle F[f], F[\varphi]\psi\rangle = \langle F[f]F[\varphi], \psi\rangle \quad \forall \psi \in S_{b_k}^{a_n}.$$

This yields the equality $F[f * \varphi] = F[f]F[\varphi]$.

It remains to show that relation (9) is true. We introduce the notation

$$I_r(\xi) := \int_{-r}^{r} \psi(\sigma) F[\varphi](\sigma) e^{i\sigma\xi} d\sigma, \quad r > 0.$$

To prove (9), it suffices to establish that $I_r(\xi) \to I(\xi)$ as $r \to +\infty$ in the space $S_{a_k}^{b_n}$, i.e., $\alpha_r(\xi) := I(\xi) - I_r(\xi) \to 0$ as $r \to +\infty$ in the topology of the space $S_{a_k}^{b_n}$. This implies that:

(i) the family of functions $\{\alpha_r^{(n)}(\xi), r > 0\}$, $n \in \mathbb{Z}_+$, uniformly converges to zero as $r \to +\infty$ on each segment $[a, b] \subset \mathbb{R}$;

(ii) $\left|\alpha_r^{(n)}\right| \leq c B^n b_n \gamma(a\xi)$ and $\left|\alpha_r^{(n)}(0)\right| \leq c B^n b_n$ $\forall n \in \mathbb{Z}_+$, where the constants $c, B, a > 0$ are independent of r.

The integral

$$\int_{-\infty}^{+\infty} D_\xi^n \left(\psi(\sigma) F[\varphi](\sigma) e^{i\sigma\xi}\right) d\sigma = \int_{-\infty}^{+\infty} (i\sigma)^n \psi(\sigma) F[\varphi](\sigma) e^{i\sigma\xi}\, d\sigma$$

converges uniformly in ξ because

$$\forall \xi \in \mathbb{R}\colon \quad \left| D_\xi^n \left(\psi(\sigma) F[\varphi](\sigma) e^{i\sigma\xi}\right)\right| \leq |\sigma^n \psi(\sigma) F[\varphi](\sigma)|, \quad \sigma \in \mathbb{R},$$

$$\int_{-\infty}^{+\infty} |\sigma^n F[\varphi](\sigma)|\, d\sigma < +\infty$$

(since $\sigma^n F[\varphi](\sigma) \in S_{b_k}^{a_n}$ for any $n \in \mathbb{Z}_+$). Thus,

$$\left|\alpha_r^{(n)}(\xi)\right| \leq \int_{|\sigma| \geq r} \left|\sigma^n F[\varphi](\sigma)\right| d\sigma \to 0, \quad r \to +\infty,$$

uniformly in $\xi \in \mathbb{R}$ as the remainder of the convergent integral. Thus, condition (i) is satisfied.

Further, we verify condition (ii). Since

$$D_\xi^n \alpha_r(\xi) \equiv D_\xi^n I(\xi) - D_\xi^n I_r(\xi), \quad n \in \mathbb{Z}_+,$$

we have

$$\left|D_\xi^n \alpha_r(\xi)\right| \leq \left|D_\xi^n I(\xi)\right| + \left|D_\xi^n I_r(\xi)\right|.$$

Consider nonnegative functions

$$D_\xi^n I_{r,+}(\xi) = \max\left(D_\xi^n I_r(\xi), 0\right), \qquad D_\xi^n I_{r,-}(\xi) = -\min\left(D_\xi^n I_r(\xi), 0\right).$$

We take into account the fact that

$$\left|D_\xi^n I_r(\xi)\right| = D_\xi^n I_{r,+}(\xi) + D_\xi^n I_{r,-}(\xi) \le 2\left|D_\xi^n I(\xi)\right|.$$

This yields

$$\left|D_\xi^n \alpha_r(\xi)\right| \le 3\left|D_\xi^n I(\xi)\right| = 3\left|D_\xi^n F[F[\varphi]\psi](\xi)\right| \quad \forall r > 0. \tag{10}$$

Since $F[F[\varphi]\psi] \in S_{a_k}^{b_n}$ $\forall \varphi \in S_{a_k}^{b_n}$, $\psi \in S_{b_k}^{a_n}$, in view of relation (10), this yields

$$\left|D_\xi^n \alpha_r(\xi)\right| \le cB^n b_n \gamma(a\xi), \quad \xi \ne 0, \qquad \left|D_\xi^n \alpha_r(0)\right| \le cB^n b_n, \quad n \in \mathbb{Z}_+,$$

where the constants $c, B, a > 0$ are independent of r. Thus, condition (ii) is satisfied.

Theorem 2 is proved.

By virtue of Theorem 2, if a generalized function f is a convolver in the space $S_{a_k}^{b_n}$, then its Fourier transform is a multiplicator in the space $S_{b_k}^{a_n}$.

4. Pseudodifferential Operators in the Generalized Spaces of Type *S*

Consider a function

$$\tilde{\gamma}(\sigma) = \begin{cases} 1, & |\sigma| \le 1, \\ \sup\limits_{k\in\mathbb{Z}_+} \dfrac{|\sigma|^k}{a_k}, & |\sigma| > 1. \end{cases}$$

It is clear that $\tilde{\gamma}$ is a nonnegative even function on $\mathbb{R}$ monotonically increasing in the interval $[1, +\infty)$, and such that $\tilde{\gamma}(\sigma) \ge 1$ for $|\sigma| \ge 1$. Since

$$\sup_{k\in\mathbb{Z}_+} \frac{|\sigma|^k}{a_k} = \frac{1}{\inf\limits_{k\in\mathbb{Z}_+} \frac{a_k}{|\sigma|^k}}, \quad \sigma \in \mathbb{R} \setminus \{0\},$$

the function $\tilde{\gamma}(\sigma)$ coincides with the function $\dfrac{1}{\gamma(\sigma)}$ for $\sigma \in \mathbb{R} \setminus \{0\}$.

Thus, if $a_k = k^{k\alpha}$, $k \in \mathbb{Z}_+$, $\alpha > 0$, then the function $\tilde{\gamma}$ constructed according to this sequence satisfies the inequalities

$$c_0 e^{\frac{\alpha}{e}|\sigma|^{1/\alpha}} \le \tilde{\gamma}(\sigma) \le e^{\frac{\alpha}{e}|\sigma|^{1/\alpha}}, \quad c_0 = e^{-\frac{\alpha e}{2}}, \quad \sigma \in \mathbb{R}.$$

By using inequality (2) satisfied by the function $\ln \gamma$ (see Lemma 2), we arrive at the following convexity inequality for the function $\ln \tilde{\gamma}$:

$$\ln \tilde{\gamma}(\sigma_1) + \ln \tilde{\gamma}(\sigma_2) \le \ln \tilde{\gamma}(\sigma_1 + \sigma_2) \quad \forall \{\sigma_1, \sigma_2\} \subset (0, +\infty). \tag{11}$$

Let φ be an infinitely differentiable function on $\mathbb{R}$ with the following properties:

(i) $\varphi(\sigma) > \Omega(\sigma) \geq \ln \tilde{\gamma}(\sigma)$, $\sigma \in \mathbb{R}$;

(ii) $\forall \varepsilon > 0\ \exists\, c_\varepsilon > 0\ \forall \sigma \in \mathbb{R}\colon \varphi(\sigma) \leq c_\varepsilon \tilde{\gamma}(\varepsilon\sigma)$;

(iii) $\exists\, c_0, B_0 > 0\ \forall n \in \mathbb{N}\ \forall \sigma \in \mathbb{R}\colon \left|\varphi^{(n)}(\sigma)\right| \leq c_0 B_0^n n!$.

Here,

$$\Omega(\sigma) = \int\limits_0^\sigma \mu(\xi)d\xi, \quad \sigma \geq 0, \quad \Omega(\sigma) = \Omega(-\sigma),$$

where $\mu(\xi)$, $0 \leq \xi < +\infty$, is a monotonically increasing continuous function such that $\mu(0) = 0$ and

$$\lim_{\xi \to +\infty} \mu(\xi) = +\infty.$$

It is clear that Ω is a continuously differentiable even function on $\mathbb{R}$ monotonically increasing on $[0, +\infty)$. The function Ω also has the following properties:

(a) $\forall \alpha \in (0,1)\ \forall \sigma \in [0, +\infty)\colon \Omega(\alpha\sigma) \leq \alpha\Omega(\sigma)$;

(b) $\forall \alpha \geq 1\ \forall \sigma \in [0, +\infty)\colon \Omega(\alpha\sigma) \geq \alpha\Omega(\sigma)$.

As an example, we prove property (a). Since the function μ is monotonically increasing on $[0, +\infty)$, we get

$$\Omega(\alpha\sigma) = \int\limits_0^{\alpha\sigma} \mu(\xi)d\xi = \alpha \int\limits_0^\sigma \mu(\alpha y)dy \leq \alpha \int\limits_0^\sigma \mu(y)dy = \alpha\Omega(\sigma).$$

Note that, by virtue of properties (ii) and (iii) of the function φ, this function is a multiplicator in the space $S^1_{a_k} \equiv S^{n^n}_{a_k}$ (the proof of this fact is similar to the proof of Lemma 3). This implies that a linear and continuous pseudodifferential operator A constructed according to the function

$$\varphi\colon A\psi = F^{-1}[\varphi(\lambda)F[\psi]] \qquad \forall \psi \in S_1^{a_n}$$

is defined in the space $S_1^{a_n} \equiv S^{a_n}_{k^k}$.

Consider a self-adjoint operator $\dfrac{i\partial}{\partial x}$ in the Hilbert space $L_2(\mathbb{R})$ with domain of definition

$$\mathcal{D}\left(\frac{i\partial}{\partial x}\right) = \{\psi \in L_2(\mathbb{R})\colon \exists\, \psi' \in L_2(\mathbb{R})\}.$$

By virtue of the main theorem on self-adjoint operators,

$$\varphi\left(\frac{i\partial}{\partial x}\right) = \int\limits_{-\infty}^{+\infty} \varphi(\lambda)\, dE_\lambda \psi,$$

where E_λ, $\lambda \in \mathbb{R}$, is the spectral function of the operator $\dfrac{i\partial}{\partial x}$. It is known (see, e.g., [18]) that

$$E_\lambda \psi = \frac{1}{2\pi} \int_{-\infty}^{\lambda} \left\{ \int_{-\infty}^{+\infty} \psi(\tau) e^{i\sigma\tau} d\tau \right\} e^{-ix\sigma} d\sigma \equiv \frac{1}{2\pi} \int_{-\infty}^{\lambda} F[\psi](\sigma) e^{-ix\sigma} d\sigma.$$

This yields the relation $dE_\lambda \psi = \dfrac{1}{2\pi} F[\psi](\lambda) e^{-ix\lambda} d\lambda$. Thus,

$$\varphi\left(\frac{i\partial}{\partial x}\right) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \varphi(\lambda) F[\psi](\lambda) e^{-ix\lambda} d\lambda = F^{-1}[\varphi(\lambda) F[\psi]] \quad \forall \psi \in S_1^{a_n},$$

i.e., the operator $\varphi\left(\dfrac{i\partial}{\partial x}\right)$ coincides with the pseudodifferential operator A in the space $S_1^{a_n}$ constructed according to the function (symbol) φ, which is a multiplicator in the space $S_{a_k}^1$.

As an example, we consider a function $\varphi(\sigma) = \left(1 + \sigma^2\right)^{1/2}$, $\sigma \in \mathbb{R}$. We can directly show that the function φ has the following properties:

(i) $\varphi \in C^\infty(\mathbb{R})$, $\varphi(\sigma) > |\sigma|$, $\sigma \in \mathbb{R}$, $\forall \varepsilon > 0 \; \exists \, c_\varepsilon > 0 \colon \varphi(\sigma) \le c_\varepsilon \exp(\varepsilon|\sigma|)$, $\sigma \in \mathbb{R}$, $c_\varepsilon = 2^{1/2} \max\{1, 1/\varepsilon\}$;

(ii) $\left|D_\sigma^n \varphi(\sigma)\right| \le c_0 B_0^n n! \le c_1 B_1^n n^n$, $n \in \mathbb{N}$, $\sigma \in \mathbb{R}$.

This implies that φ is a multiplicator in the space $S_1^1 \equiv S_{k^k}^{n^n}$. Hence, the operator

$$\varphi\left(\frac{i\partial}{\partial x}\right) = \sqrt{I + \left(\frac{i\partial}{\partial x}\right)^2} = \sqrt{I - \left(\frac{\partial}{\partial x}\right)^2} = \int_{-\infty}^{+\infty} \left(1 + \lambda^2\right)^{1/2} dE_\lambda$$

coincides with the pseudodifferential operator $F^{-1}\left[(1 + \lambda^2)^{1/2} F\right]$ in the space S_1^1.

A function

$$\varphi(\sigma) = (1 + \sigma^2)^{\omega/2}, \quad \sigma \in \mathbb{R},$$

where $\omega \in [1, 2)$, is a fixed parameter, has similar properties:

$$\varphi \in C^\infty(\mathbb{R}), \quad \varphi(\sigma) > |\sigma|^\omega, \quad \sigma \in \mathbb{R}, \quad \varphi(\sigma) \le c_\varepsilon \exp\left\{\varepsilon|\sigma|^\omega\right\}, \quad \sigma \in \mathbb{R},$$

where $c_\varepsilon = 2^{\omega/2} \max\{1, 1/\varepsilon\}$, and

$$\left|D_\sigma^n \varphi(\sigma)\right| \le c_0 B_0^n n! \le c_1 B_1^n n^n.$$

The function φ is a multiplicator in the space $S_{1/\omega}^1 \equiv S_{k^{k/\omega}}^{n^n}$ and the operator

$$\varphi\left(\frac{i\partial}{\partial x}\right) = \left(I - \left(\frac{\partial^2}{\partial x^2}\right)\right)^{\omega/2}$$

coincides in the space $S_1^{1/\omega}$ with the pseudodifferential operator constructed according to the symbol

$$\varphi(\sigma) = \left(1+\sigma^2\right)^{\omega/2}, \quad \sigma \in \mathbb{R}.$$

5. Nonlocal (in Time) Problem

Consider a differential-operator equation

$$\frac{\partial u(t,x)}{\partial t} + \varphi\left(\frac{i\partial}{\partial x}\right) u(t,x) = 0, \quad (t,x) \in (0,+\infty)\times\mathbb{R} \equiv \Omega_+, \tag{12}$$

where $\varphi\left(\frac{i\partial}{\partial x}\right)$ is understood as a pseudodifferential operator in the space $S_1^{a_n}$ constructed according to the function φ, which is a multiplicator in the space $S_{a_k}^1$ [the function φ has properties (i)–(iii) formulated in Sec. 4]. Hence,

$$\varphi\left(\frac{i\partial}{\partial x}\right)\psi = F^{-1}\left[\varphi(\sigma)F[\psi]\right] \quad \forall \psi \in S_1^{a_n}.$$

A solution of Eq. (12) is understood as a function $u(t,x)$, $(t,x) \in \Omega_+$, continuously differentiable with respect to the variable t and such that $u(\cdot,x) \in S_1^{a_n}$ for every $t > 0$ and $u(t,x)$, $(t,x) \in \Omega_+$, satisfies Eq. (12).

For Eq. (12), we formulate a nonlocal multipoint (in time) problem as follows:

To find a solution of Eq. (12) satisfying the condition

$$\mu u(0,x) - \mu_1 u(t_1,x) - \ldots - \mu_m u(t_m,x) = f(x), \quad x \in \mathbb{R}, \quad f \in S_1^{a_n}, \tag{13}$$

where

$$u(0,x) = \lim_{t\to+0} u(t,x), \quad m \in \mathbb{N},$$

$$\{\mu,\mu_1,\ldots,\mu_m\} \subset (0,+\infty), \quad \{t_1,\ldots,t_m\} \subset (0,+\infty)$$

are fixed numbers such that $0 < t_1 < \ldots < t_m < +\infty$ and, moreover,

$$\mu > m\sum_{k=1}^{m}\mu_k.$$

By using the Fourier transforms, we seek the solution of problem (12), (13) in the form $u(t,x) = F^{-1}[v(t,\cdot)]$. For the function $v\colon \Omega_+ \to \mathbb{R}$, we obtain the following problem with the parameter σ:

$$\frac{dv(t,\sigma)}{dt} + \varphi(\sigma)v(t,\sigma) = 0, \quad (t,\sigma) \in \Omega_+, \tag{14}$$

$$\mu v(0,\sigma) - \sum_{k=1}^{m}\mu_k v(t_k,\sigma) = \tilde{f}(\sigma), \quad \sigma \in \mathbb{R}, \tag{15}$$

where $\tilde{f}(\sigma) = F[f](\sigma)$. The general solution of Eq. (14) has the form

$$v(t,\sigma) = c\exp\{-t\varphi(\sigma)\}, \quad (t,x) \in \Omega_+, \tag{16}$$

where $c = c(\sigma)$ is determined from condition (15). Substituting (16) in Eq. (15), we obtain

$$c = \tilde{f}(\sigma)\left(\mu - \sum_{k=1}^{m} \mu_k \exp\{-t_k\varphi(\sigma)\}\right)^{-1}, \quad \sigma \in \mathbb{R}.$$

We introduce the notation: $G(t,x) = F^{-1}[Q(t,\sigma)]$ and $Q(t,\sigma) = Q_1(t,\sigma)Q_2(\sigma)$, where

$$Q_1(t,\sigma) = \exp\{-t\varphi(\sigma)\}, \qquad Q_2(\sigma) = \left(\mu - \sum_{k=1}^{m} \mu_k Q_1(t_k,\sigma)\right)^{-1}.$$

Further, as a result of formal reasoning, we arrive at the relation

$$u(t,x) = \int_{\mathbb{R}} G(t,x-\xi)f(\xi)d\xi = G(t,x) * f(x).$$

Indeed,

$$u(t,x) = (2\pi)^{-1}\int_{\mathbb{R}} Q(t,\sigma)\left(\int_{\mathbb{R}} f(\xi)e^{i\sigma\xi}d\xi\right)e^{-i\sigma x}d\sigma$$

$$= \int_{\mathbb{R}}\left((2\pi)^{-1}\int_{\mathbb{R}} Q(t,\sigma)e^{-i\sigma(x-\xi)}d\xi\right)f(\xi)d\xi$$

$$= \int_{\mathbb{R}} G(t,x-\xi)f(\xi)d\xi = G(t,x) * f(x), \quad (t,x) \in \Omega_+.$$

The validity of the performed transformations and the property of convergence of the corresponding integrals follow from the properties of the function G presented in what follows. The properties of the function G are determined by the properties of the function Q because $G = F^{-1}[Q]$. Thus, first of all, we study the properties of the function $Q(t,\sigma)$ regarded as a function of the argument σ.

Lemma 4. *The derivatives of the function $Q_1(t,\sigma)$, $(t,\sigma) \in \Omega_+$ (with respect to the variable σ) satisfy the following estimates:*

$$\left|D_\sigma^n Q_1(t,\sigma)\right| \le cA^n t^{\omega n} n^n \exp\{-t\varphi(\sigma)\}, \quad (t,\sigma) \in \Omega_+, \quad n \in \mathbb{N}, \tag{17}$$

where $\omega = 0$ for $0 < t \le 1$, $\omega = 1$ for $t > 1$, and the constants $c > 1$ and $A > 0$ are independent of t.

Proof. To prove the assertion, we use the Faa di Bruno formula of differentiation of a composite function

$$D_\sigma^n F(g(\sigma)) = \sum_{p=1}^{n} \frac{d^p}{dg^p} F(g) \sum \frac{n!}{p_1! \ldots p_l!} \left(\frac{d}{d\sigma} g(\sigma)\right)^{p_1} \ldots \left(\frac{1}{l!} \frac{d^l}{d\sigma^l} g(\sigma)\right)^{p_l}, \quad n \in \mathbb{N}$$

(the index of summation runs through all integer nonnegative solutions of the equation $p_1 + 2p_2 + \ldots + lp_l = n$, $p_1 + \ldots + p_l = p$), where we set $F = e^g$ and $g = -t\varphi(\sigma)$. Then

$$D_\sigma^n e^{-t\varphi(\sigma)} = e^{-t\varphi(\sigma)} \sum_{p=1}^{n} \sum \frac{n!}{p_1! \ldots p_l!} \Lambda,$$

where the symbol Λ denotes the expression

$$\Lambda := \left(\frac{d}{d\sigma} (-t\varphi(\sigma))\right)^{p_1} \left(\frac{1}{2!} \frac{d^2}{d\sigma^2} (-t\varphi(\sigma))\right)^{p_2} \ldots \left(\frac{1}{l!} \frac{d^l}{d\sigma^l} (-t\varphi(\sigma))\right)^{p_l}.$$

In view of property (iii) of the function φ, which remains true for the derivatives of the function φ, we obtain

$$|\Lambda| \le c_0^{p_1+\ldots+p_l} B_0^{p_1+2p_2+\ldots+lp_l} t^{p_1+\ldots+p_l} \le \tilde{c}_0^n t^p B_0^n, \quad \tilde{c}_0 = \max\{1, c_0\}.$$

By using property (i) of the function φ and the Stirling formula, we arrive at the inequalities

$$\left|D_\sigma^n Q_1(t,\sigma)\right| \le \tilde{c}_0^n B_0^n t^{\omega n} n! \exp\{-t\varphi(\sigma)\} \le c A^n t^{\omega n} n^n \exp\{-t\varphi(\sigma)\}, \quad \sigma \in \mathbb{R}, \tag{18}$$

where $\omega = 0$ for $0 < t \le 1$, $\omega = 1$ for $t > 1$, and the constants $c > 1$ and $A > 0$ are independent of t.

Lemma 4 is proved.

Remark 1. It follows from estimates (18) that $Q_1(t,\cdot) \in S_{a_k}^1$ for any $t > 0$.

Indeed, let $t \in (0,1)$. By using property (i) of the function φ and property (a) of the function Ω, we obtain the inequalities

$$e^{-t\varphi(\sigma)} \le e^{-t\Omega(\sigma)} \le e^{-\Omega(t\sigma)} \le e^{-\ln\tilde{\gamma}(t\sigma)} = e^{\ln\gamma(t\sigma)} = \gamma(t\sigma), \quad \sigma \in \mathbb{R} \setminus \{0\}$$

$\left(\text{here, we have used the fact that } \tilde{\gamma}(\sigma) = \dfrac{1}{\gamma(\sigma)}, \sigma \in \mathbb{R} \setminus \{0\}\right)$.

If $t > 1$, $t \ne n$, where $n \in \{2, 3, \ldots\}$, then $t = [t] + \{t\}$. We get

$$e^{-t\varphi(\sigma)} = e^{-[t]\varphi(\sigma)} e^{-\{t\}\varphi(\sigma)} \le e^{-\{t\}\varphi(\sigma)} \le e^{-\{t\}\Omega(\sigma)} \le e^{-\Omega(\{t\}\sigma)}$$

$$\le e^{-\ln\tilde{\gamma}(\{t\}\sigma)} = \gamma(\{t\}\sigma), \quad \sigma \in \mathbb{R} \setminus \{0\}.$$

If $t = n$, $n \in \{2, 3, \ldots\}$, then $t = 1 + n - 1$. In this case, we obtain

$$e^{-t\varphi(\sigma)} = e^{-\varphi(\sigma)} e^{-(n-1)\varphi(\sigma)} \le e^{-\varphi(\sigma)} \le e^{-\Omega(\sigma)} \le e^{-\ln\tilde{\gamma}(\sigma)} = \gamma(\sigma), \quad \sigma \in \mathbb{R} \setminus \{0\}.$$

Hence,

$$e^{-t\varphi(\sigma)} \le \gamma(a\sigma), \quad \sigma \in \mathbb{R} \setminus \{0\}, \tag{19}$$

where $a = \{t\}$ for $t \ne n$, $n \in \mathbb{N}$, and $a = 1$ for $t = n$, $n \in \mathbb{N}$.

Therefore, for fixed $t > 0$, the inequalities

$$\left|D_\sigma^n Q_1(t,\sigma)\right| \le c\tilde{A}^n n^n \gamma(a\sigma), \quad \sigma \in \mathbb{R} \setminus \{0\}, \quad n \in \mathbb{Z}_+, \tag{20}$$

where $\tilde{A} = At^{\omega}$, are true. Since $e^{-t\varphi(0)} \le e^{-t\Omega(0)} = 1$, we have

$$D_\sigma^n Q_1(t,0) \le c\tilde{A}^n n^n, \quad n \in \mathbb{Z}_+.$$

By using this result and Lemma 2, we conclude that $Q_1(t,\sigma) \in S_{a_k}^1$ for any $t > 0$.

Lemma 5. *The function Q_2 is a multiplicator in the space $S_{a_k}^1$.*

Proof. By using property (i) of the function φ, we conclude that the inequalities

$$Q_1(t_k,\sigma) \le \exp\{-t_k\varphi(\sigma)\} < 1, \quad k \in \{1,\dots,m\}, \quad \sigma \in \mathbb{R},$$

are true. Since $\mu > \sum_{k=1}^m \mu_k$, we get

$$\frac{1}{\mu}\sum_{k=1}^m \mu_k Q_1(t_k,\sigma) < \frac{1}{\mu}\sum_{k=1}^m \mu_k < 1.$$

By using the polynomial formula, we obtain

$$Q_2(\sigma) = \frac{1}{\mu}\left(1 - \frac{1}{\mu}\sum_{k=1}^m \mu_k Q_1(t_k,\sigma)\right)^{-1} = \frac{1}{\mu}\sum_{r=0}^{\infty}\mu^{-r}\left(\sum_{k=1}^m \mu_k e^{-t_k\varphi(\sigma)}\right)^r$$

$$= \sum_{r=0}^{\infty}\mu^{-(r+1)}\sum_{r_1+\ldots+r_m=r}\frac{r!}{r_1!\ldots r_m!}\left(\mu_1 e^{-t_1\varphi(\sigma)}\right)^{r_1}\ldots\left(\mu_m e^{-t_m\varphi(\sigma)}\right)^{r_m}$$

$$= \sum_{r=0}^{\infty}\mu^{-(r+1)}\sum_{r_1+\ldots+r_m=r}\frac{r!}{r_1!\ldots r_m!}\mu_1^{r_1}\ldots\mu_m^{r_m}Q_1(\lambda,\sigma),$$

where $\lambda := t_1r_1 + \ldots + t_mr_m$ and $Q_1(\lambda,\sigma) = e^{-\lambda\varphi(\sigma)}$. By using this result and (17), we arrive at the inequalities

$$\left|D_\sigma^n Q_2(\sigma)\right| \le cA^n n^n \sum_{r=0}^{\infty}\mu^{-(r+1)}\sum_{r_1+\ldots+r_m=r}\frac{r!}{r_1!\ldots r_m!}\mu_0^r\lambda^{\omega n}\exp\{-\lambda\varphi(\sigma)\}$$

$$\leq cA^n t_m^{\omega n} n^n \sum_{r=0}^{\infty} \mu^{-(r+1)} \mu_0^r r^n \sum_{r_1+\ldots+r_m=r} \frac{r!}{r_1!\ldots r_m!}, \quad n \in \mathbb{N},$$

where $\mu_0 = \max\{\mu_1, \ldots \mu_m\}$. Further,in view of the fact that

$$\sum_{r_1+\ldots+r_m=r} \frac{r!}{r_1!\ldots r_m!} = m^r,$$

we get

$$\left|D_\sigma^n Q_2(\sigma)\right| \leq c' A_1^n n^n \sum_{r=0}^{\infty} \tilde{\mu}^r r^n = \tilde{c} A_1^n n^n, \quad n \in \mathbb{N}, \tag{21}$$

where $\tilde{\mu} = \mu^{-1}\mu_0 m < 1$, $c' = c\mu^{-1}$,

$$\tilde{c} = c' \sum_{r=0}^{\infty} \tilde{\mu}^r r^n,$$

and $A_1 = At_m^\omega$. The last inequality and the boundedness of the function Q_2 on $\mathbb{R}$ imply that Q_2 is a multiplicator in the space $S_{a_k}^1$.

Lemma 5 is proved.

By using (17), (21) and the Leibniz formula of differentiation of the product of two functions, we find

$$\left|D_\sigma^n Q(t,\sigma)\right| = \left|\sum_{l=0}^{n} C_n^l D_\sigma^l Q_1(t,\sigma) D_\sigma^{n-l} Q_2(\sigma)\right|$$

$$\leq c\tilde{c} \sum_{l=0}^{n} C_n^l A^l t^{\omega l} l^l A_1^{n-l} (n-l)^{n-l} \exp\{-t\varphi(\sigma)\}$$

$$\leq \tilde{b}\tilde{B}^n t^{\omega n} n^n \exp\{-t\varphi(\sigma)\},$$

where $\tilde{b} = c\tilde{c}$ and $\tilde{B} = 2\max\{A, A_1\}$. By using the last inequality, Remark 1, and estimate (20), we conclude that $Q(t,\sigma)$ is an element of the space $S_{a_k}^1$ as a function of the variable σ (for any $t > 0$). Since $G = F^{-1}[Q]$, the function $G(t,\cdot)$ is an element of the space $S_1^{a_n}$ for any $t > 0$.

Lemma 6. *The function $G(t,\cdot)$, $t \in (0,+\infty)$, as an abstract function of the parameter t with values in the space $S_1^{a_n}$, is differentiable with respect to t.*

Proof. In view of the continuity of direct and inverse Fourier transforms in the spaces of type S, in order to prove the required assertion, it suffices to show that the function $F[G(t,\cdot)] = Q(t,\cdot)$, regarded as an abstract function of the parameter t with values in the space $S_{a_k}^1$, is differentiable with respect to t. In other words, it is necessary to prove that the limit relation

$$\Phi_{\Delta t}(\sigma) := \frac{1}{\Delta t}\left[Q(t+\Delta t,\cdot) - Q(t,\cdot)\right] \to \frac{\partial}{\partial t} Q(t,\cdot), \quad \Delta t \to 0,$$

is true in a sense that:

(i) $D_\sigma^n \Phi_{\Delta t}(\sigma) \underset{\Delta t \to 0}{\longrightarrow} D_\sigma^n(-\varphi(\sigma) Q(t,\sigma))$, $n \in \mathbb{Z}_+$, uniformly on each segment $[a,b] \subset \mathbb{R}$;

(ii) $\left|D_\sigma^n \Phi_{\Delta t}(\sigma)\right| \le \overline{c}\,\overline{B}^n n^n \gamma(\overline{a}\sigma)$, $n \in \mathbb{Z}_+$, $\sigma \in \mathbb{R} \setminus \{0\}$,

where the constants $\overline{c}$, $\overline{a}$, $\overline{B} > 0$ are independent of Δt if Δt is sufficiently small.

The function $Q(t,\sigma)$, $(t,\sigma) \in \Omega_+$, is differentiable with respect to t in the ordinary sense. By virtue of the Lagrange theorem on finite increments, we get

$$\Phi_{\Delta t}(\sigma) = -\varphi(\sigma) Q(t + \theta \Delta t, \sigma), \quad 0 < \theta < 1.$$

Hence,

$$D_\sigma^n \Phi_{\Delta t}(\sigma) = -\sum_{l=0}^{n} C_n^l D_\sigma^l \varphi(\sigma) D_\sigma^{n-l} Q(t + \theta \Delta t, \sigma) \tag{22}$$

and

$$D_\sigma^n \left(\Phi_{\Delta t}(\sigma) - \frac{\partial}{\partial t} Q(t,\sigma)\right) = -\sum_{l=0}^{n} C_n^l D_\sigma^l \varphi(\sigma) \left[D_\sigma^{n-l} Q(t + \theta \Delta t, \sigma) - D_\sigma^{n-l} Q(t,\sigma)\right].$$

Since

$$D_\sigma^{n-l} Q(t + \theta \Delta t, \sigma) - D_\sigma^{n-l} Q(t,\sigma) = D_\sigma^{n-l+1} Q(t + \theta_1 \Delta t, \sigma) \theta \Delta t, \quad 0 < \theta_1 < 1,$$

in view of estimates (17), this yields

$$D_\sigma^{n-l+1} Q(t + \theta_1 \Delta t, \sigma) \theta \Delta t \to 0, \quad \Delta t \to 0,$$

uniformly on any segment $[a,b] \subset \mathbb{R}$. By using the properties of the function φ, we obtain

$$D_\sigma^n \Phi_{\Delta t}(\sigma) \to D_\sigma^n \left(\frac{\partial}{\partial t} Q(t,\sigma)\right), \quad \Delta t \to 0,$$

uniformly on any segment $[a,b] \subset \mathbb{R}$. Thus, condition (i) is satisfied.

By using (22) and estimates for the functions $\varphi(\sigma)$ and $Q(t,\sigma)$ and their derivatives, we get

$$\left|D_\sigma^n \Phi_{\Delta t}(\sigma)\right| \le \tilde{c} \sum_{l=0}^{n} C_n^l B^l l^l A^{n-l} (t + \theta \Delta t)^{\omega(n-l)} e^{-(t+\theta \Delta t)\varphi(\sigma)} \tilde{\gamma}(\varepsilon\sigma)$$

(here, $\varepsilon > 0$ is an arbitrary fixed number and $\tilde{c} = \tilde{c}(\varepsilon) > 0$). Since $t + \theta \Delta t \le T$, where $T > 1$ is fixed, we arrive at the estimate

$$\left|D_\sigma^n \Phi_{\Delta t}(\sigma)\right| \le \overline{c}\,\overline{L}^n n^n e^{-t\varphi(\sigma)} \tilde{\gamma}(\varepsilon\sigma), \quad \overline{L} = 2 \max\{AT, B\}.$$

Further, by using the inequality

$$\exp\{-t\varphi(\sigma)\} \le \gamma(a\sigma), \quad \sigma \in \mathbb{R} \setminus \{0\}$$

[see (19)], we obtain

$$\begin{aligned}\left|D_\sigma^n \Phi_{\Delta t}(\sigma)\right| &\le \tilde{c}\tilde{L}^n n^n \gamma(a\sigma)\tilde{\gamma}(\varepsilon\sigma)\\ &== \tilde{c}\tilde{L}^n n^n e^{-\ln\tilde{\gamma}(a\sigma)} e^{\ln\tilde{\gamma}(\varepsilon\sigma)}\\ &= \tilde{c}\tilde{L}^n n^n e^{-\ln\tilde{\gamma}(a\sigma)+\ln\tilde{\gamma}(\varepsilon\sigma)}, \quad \sigma \in \mathbb{R}\setminus\{0\},\end{aligned}$$

where $\tilde{\gamma}(\sigma) = 1/\gamma(\sigma)$). By using the convexity inequality (11) satisfied by the function $\ln\tilde{\gamma}$, we arrive at the inequality

$$\ln\tilde{\gamma}(\varepsilon\sigma) + \ln\tilde{\gamma}((a-\varepsilon)\sigma) \le \ln\tilde{\gamma}(a\sigma), \quad \varepsilon \in (0,a), \quad \sigma \in \mathbb{R}\setminus\{0\},$$

or at the inequality

$$-\ln\tilde{\gamma}(a\sigma) + \ln\tilde{\gamma}(\varepsilon\sigma) \le -\ln\tilde{\gamma}((a-\varepsilon)\sigma), \quad \sigma \in \mathbb{R}\setminus\{0\}, \quad \varepsilon \in (0,a).$$

Thus,

$$\left|D_\sigma^n \Phi_{\Delta t}(\sigma)\right| \le \tilde{c}\tilde{L}^n n^n e^{-\ln\tilde{\gamma}(a_1\sigma)} = \tilde{c}\tilde{L}^n n^n \gamma(a_1\sigma),$$

$$a_1 = a - \varepsilon, \quad n \in \mathbb{Z}_+, \quad \sigma \in \mathbb{R}\setminus\{0\}.$$

moreover, all constants are independent of Δt, i.e., condition (ii) is satisfied.

Lemma 6 is proved.

Corollary 1. *The following relation is true:*

$$\frac{\partial}{\partial t}\big(f * G(t,x)\big) = f * \frac{\partial G(t,x)}{\partial t} \quad \forall f \in \big(S_1^{a_n}\big)', \quad t > 0.$$

Proof. According to the definition of convolution of a generalized function with test function, we get

$$f * G(t,x) = \left\langle f_\xi, T_{-x}\check{G}(t,\xi)\right\rangle, \quad \check{G}(t,\xi) = G(t,-\xi).$$

Thus,

$$\begin{aligned}\frac{\partial}{\partial t}\,(f * G(t,\cdot)) &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\,[f * G(t+\Delta t,\cdot) - f * G(t,\cdot)]\\ &= \lim_{\Delta t\to 0}\left\langle f_\xi, \frac{1}{\Delta t}\left[T_{-x}\check{G}(t+\Delta t,\cdot) - T_{-x}\check{G}(t,\cdot)\right]\right\rangle.\end{aligned}$$

By Lemma 6, the limit relation

$$\frac{1}{\Delta t}\left[T_{-x}\check{G}(t+\Delta t,\xi)-T_{-x}\check{G}(t,\xi)\right]\to\frac{\partial}{\partial t}T_{-x}\check{G}(t,\xi),\quad \Delta t\to 0,$$

is true in a sense of convergence in the topology of the space $S_1^{a_n}$. Thus, we obtain

$$\frac{\partial}{\partial t}\big(f*G(t,\cdot)\big)=\left\langle f_\xi,\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[T_{-x}\check{G}(t+\Delta t,\xi)-T_{-x}\check{G}(t,\xi)\right]\right\rangle$$

$$=\left\langle f_\xi,\frac{\partial}{\partial t}T_{-x}\check{G}(t,\xi)\right\rangle=\left\langle f_\xi,T_{-x}\frac{\partial}{\partial t}\check{G}(t,\xi)\right\rangle=f*\frac{\partial}{\partial t}G(t,\cdot).$$

Corollary 1 is proved.

Lemma 7. *In the space $\big(S_1^{a_n}\big)'$, the following limit relations are true:*

(i) $G(t,\cdot)\to F^{-1}[Q_2]$, $\Delta t\to +0$.

(ii)

$$\mu G(t,\cdot)-\sum_{k=1}^{m}\mu_k G(t_k,\cdot)\to\delta,\quad t\to +0, \tag{23}$$

where δ is the Dirac delta function.

Proof. (i) In view of the continuity of direct and inverse Fourier transforms in the spaces of type S', in order to prove the required assertion, it suffices to show that

$$F[G(t,\cdot)]=Q(t,\cdot)=Q_1(t,\cdot)Q_2(\cdot)\to Q_2(\cdot),\quad t\to +0,$$

in the space $\big(S_{a_k}^1\big)'$. To this end, we take an arbitrary function $\psi\in S_{a_k}^1$, use the fact that Q_2 is a multiplicator in the space $S_{a_k}^1$, and apply the Lebesgue theorem on the limit transition in the Lebesgue integral. As a result, we obtain

$$\langle Q_1(t,\cdot)Q_2(\cdot),\psi\rangle=\langle Q_1(t,\cdot),Q_2(\cdot)\psi(\cdot)\rangle$$

$$=\int_{\mathbb{R}}Q_1(t,\sigma)Q_2(\sigma)\psi(\sigma)d\sigma\underset{t\to+0}{\longrightarrow}\int_{\mathbb{R}}Q_2(\sigma)\psi(\sigma)d\sigma$$

$$=\langle 1,Q_2(\cdot)\psi(\cdot)\rangle=\langle Q_2,\psi\rangle.$$

This yields Assertion 1 of Lemma 7.

(ii) By using assertion 1, we get

$$\mu G(t,\cdot) - \sum_{k=1}^{m} \mu_k G(t_k,\cdot) \underset{t\to+0}{\longrightarrow} \mu F^{-1}[Q_2] - \sum_{k=1}^{m} \mu_k G(t_k,\cdot)$$

$$= \mu F^{-1}[Q_2] - \sum_{k=1}^{m} \mu_k F^{-1}[Q_1(t_k,\cdot)Q_2(\cdot)]$$

$$= F^{-1}\left[\mu Q_2 - \sum_{k=1}^{m} \mu_k Q_1(t_k,\cdot)Q_2(\cdot)\right] = F^{-1}\left[\left(\mu - \sum_{k=1}^{m} \mu_k Q_1(t_k,\cdot)\right) Q_2(\cdot)\right]$$

$$= F^{-1}\left[\left(\mu - \sum_{k=1}^{m} \mu_k Q_1(t_k,\sigma)\right)\left(\mu - \sum_{k=1}^{m} \mu_k Q_1(t_k,\sigma)\right)^{-1}\right] = F^{-1}[1] = \delta.$$

Thus, relation (23) is true in the space $\left(S_1^{a_n}\right)'$.
Lemma 7 is proved,

Remark 2. If $\mu = 1$ and $\mu_1 = \ldots = \mu_m = 0$, then problem (12), (13) is reduced to the Cauchy problem for Eq. (12). In this case,

$$Q_2(\sigma) = 1 \quad \forall \sigma \in \mathbb{R}, \qquad G(t,x) = F^{-1}\left[e^{-t\varphi(\sigma)}\right],$$

and

$$G(t,\cdot) \to F^{-1}[1] = \delta, \quad t \to +0,$$

in the space $\left(S_1^{a_n}\right)'$.

Theorem 3. *Suppose that*

$$\omega(t,x) = f * G(t,x), \quad f \in \left(S_{1,*}^{a_n}\right)', \quad (t,x) \in \Omega_+,$$

where $\left(S_{1,*}^{a_n}\right)'$ *is a class of convolvers in the space* $S_1^{a_n}$. *Then the limit relation*

$$\mu\omega(t,\cdot) - \sum_{k=1}^{m} \mu_k \omega(t_k,\cdot) \to f, \quad t \to +0, \tag{24}$$

is true in the space $\left(S_1^{a_n}\right)'$.

Proof. We now prove that the limit relation

$$F\left[\mu\omega(t,\cdot) - \sum_{k=1}^{m} \mu_k \omega(t_k,\cdot)\right] \to F[f], \quad t \to +0, \tag{25}$$

is true in the space $\left(S_{a_k}^1\right)'$. Since $f \in \left(S_{1,*}^{a_n}\right)'$ and $G(t,\cdot) \in S_1^{a_n}$ for any $t > 0$, we find (see Theorem 2)

$$F[\omega(t,\cdot)] = F[f * G(t,\cdot)] = F[f]F[G(t,\cdot)] = F[f]Q(t,\cdot).$$

Thus, it is necessary to show that

$$F[f]\left(\mu Q(t,\cdot) - \sum_{k=1}^{m} \mu_k Q(t_k,\cdot)\right) \to F[f]$$

as $t \to +0$ in the space $\left(S^1_{a_k}\right)'$. Since

$$Q(t,\cdot) = Q_1(t,\cdot)Q_2(\cdot) \to Q_2(\cdot) \quad \text{as} \quad t \to +0$$

in the space $\left(S^1_{a_k}\right)'$ [see the proof of Assertion (i) in Lemma 7)], we conclude that

$$\mu Q(t,\cdot) - \sum_{k=1}^{m} \mu_k Q(t_k,\cdot) \underset{t\to+0}{\longrightarrow} \mu Q_2(\cdot) - \sum_{k=1}^{m} \mu_k Q_1(t_k,\cdot)Q_2(\cdot)$$

$$= \left(\mu - \sum_{k=1}^{m} \mu_k Q_1(t_k,\sigma)\right) Q_2(\sigma) = 1$$

in the space $\left(S^1_{a_k}\right)'$. Thus, relations (25) and (24) are true in the corresponding spaces.

Theorem 3 is proved.

The function $G(t,x)$, $(t,x) \in \Omega_+$, satisfies Eq. (12). Indeed,

$$\frac{\partial}{\partial t} G(t,x) = \frac{\partial}{\partial t} F^{-1}[Q(t,\sigma)] = F^{-1}\left[\frac{\partial}{\partial t} Q(t,\sigma)\right] = -F^{-1}[\varphi(\sigma)Q(t,\sigma)],$$

$$\varphi\left(\frac{i\partial}{\partial x}\right) G(t,x) = F^{-1}\big[\varphi(\sigma)F^{-1}[G(t,\cdot)]\big] = F^{-1}[\varphi(\sigma)Q(t,\sigma)].$$

Thus,

$$\frac{\partial G(t,x)}{\partial t} + \varphi\left(\frac{i\partial}{\partial x}\right) G(t,x) = 0, \quad (t,x) \in \Omega_+,$$

Q.E.D.

By Theorem 3, the nonlocal multipoint (in time) problem for Eq. (12) can be formulated as follows:

To find a function $u(t,x)$, $(t,x) \in \Omega_+$, satisfying Eq. (12) and the condition

$$\mu \lim_{t\to+0} u(t,\cdot) - \sum_{k=1}^{m} \mu_k u(t_k,\cdot) = f, \quad f \in \left(S^{a_n}_{1,*}\right)' \tag{26}$$

[the limit relation (26) is considered in the space $\left(S^{a_n}_1\right)'$, and the restrictions imposed on the parameters μ, $\mu_1,\ldots,\mu_m, t_1,\ldots,t_m$ are the same as for problem (12), (13)].

Theorem 4. *The nonlocal multipoint (in time) problem (12), (13) is correctly solvable and its solution is given by the formula*

$$u(t,x) = f * G(t,x), \quad (t,x) \in \Omega_+,$$

where $u(t,\cdot) \in S_1^{a_n}$ *for any* $t > 0$.

Proof. We now show that the function $u(t,x)$, $(t,x) \in \Omega_+$, satisfies Eq. (12). Indeed (see Corollary 1),

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial}{\partial t}(f * G(t,\cdot)) = f * \frac{\partial G(t,\cdot)}{\partial t}$$

and

$$\varphi\left(\frac{i\partial}{\partial x}\right) u(t,x) = F^{-1}[\varphi(\sigma)F[f * G(t,\cdot)]].$$

Since f is a convolver in the space $S_1^{a_n}$, we have

$$F[f * G(t,\cdot)] = F[f]F[G(t,\cdot)] = F[f]Q(t,\cdot).$$

Hence,

$$\begin{aligned} \varphi\left(\frac{i\partial}{\partial x}\right) u(t,x) &= F^{-1}\big[\varphi(\sigma)Q(t,\sigma)F[f](\sigma)\big] \\ &= -F^{-1}\left[\frac{\partial}{\partial t} Q(t,\sigma)F[f]\right] = -F^{-1}\left[F\left[\frac{\partial G(t,\cdot)}{\partial t}\right]F[f]\right] \\ &= -F^{-1}\left[F\left[f * \frac{\partial G(t,\cdot)}{\partial t}\right]\right] = -f * \frac{\partial G(t,\cdot)}{\partial t}. \end{aligned}$$

This implies that the function $u(t,x)$, $(t,x) \in \Omega_+$, satisfies Eq. (12).

By Theorem 3, the function $u(t,x)$, $(t,x) \in \Omega_+$, satisfies condition (26) in the indicated sense.

It remains to show that problem (12), (26) is uniquely solvable. To this end, we consider a Cauchy problem

$$\frac{\partial v}{\partial t} - Av = 0, \quad (t,x) \in [0,t_0) \times \mathbb{R} \equiv \Omega', \quad 0 \le t < t_0 < +\infty, \tag{27}$$

$$v(t,\cdot)|_{t=t_0} = \psi, \quad \psi \in \big(S_{1,*}^{a_n}\big)', \tag{28}$$

where A is the restriction of the operator adjoint to the operator

$$\varphi\left(\frac{i\partial}{\partial x}\right) = F^{-1}[\varphi F]$$

to the space $S_1^{a_n} \subset \big(S_{1,*}^{a_n}\big)'$. Condition (28) is understood in a weak sense. The Cauchy problem (27), (28) is solvable and, moreover, $v(t,\cdot) \in S_1^{a_n}$ for any $t \in [0,t_0)$ $\big[$the Cauchy problem (27), (28) is investigated by using the same scheme as in the investigation of problem (10), (26) with the parameters $\mu = 1$, $\mu_1 = \ldots = \mu_m = 0$ by taking into account the fact that $A = F^{-1}[\varphi \cdot F]$ and, moreover,

$$v(t,x) = \psi * \tilde{G}(t,x) \quad \text{and} \quad \tilde{G}(t,x) = F^{-1}\big[e^{(t-t_0)\varphi(\sigma)}\big]\big].$$

Let $Q_{t_0}^t \colon \big(S_{1,*}^{a_n}\big)' \to S_1^{a_n}$ be an operator that associates a function $\psi \in \big(S_{1,*}^{a_n}\big)'$ with the solution of problem (27), (28). The operator $Q_{t_0}^t$ is linear and continuous. Furthermore, it is defined for any t and t_0 such that $0 \le t < t_0 < +\infty$ and has the following properties;

$$\forall \psi \in \big(S_{1,*}^{a_n}\big)' \colon \; \frac{dQ_{t_0}^t \psi}{dt} - AQ_{t_0}^t \psi = 0, \quad \lim_{t \to t_0} Q_{t_0}^t \psi = \psi$$

(the limit is considered in the space $\big(S_1^{a_n}\big)'$).

Consider a solution $u(t,x)$, $(t,x) \in \Omega_+$, of problem (12), (26), regarded as a regular functional from the space $\big(S_{1,*}^{a_n}\big)'$. We prove that problem (12), (26) may have only one solution in the space $\big(S_{1,*}^{a_n}\big)'$. To this end, it suffices to show that the functional $u(t,x) \equiv 0$ is the sole possible solution of Eq. (12) with the trivial initial conditions (for any $t \in (0, +\infty)$). Further, we apply the functional u to the function $Q_{t_0}^t \psi$, where ψ is an arbitrarily fixed element of the space $S_1^{a_n} \subset \big(S_{1,*}^{a_n}\big)'$. Differentiating with respect to t and using Eqs. (12) and (27), we obtain

$$\begin{aligned}
\frac{\partial}{\partial t} \big\langle u(t,\cdot), Q_{t_0}^t \psi \big\rangle &= \left\langle \frac{\partial u}{\partial t}, Q_{t_0}^t \psi \right\rangle + \left\langle u, \frac{\partial Q_{t_0}^t \psi}{\partial t} \right\rangle \\
&= -\left\langle \varphi\left(\frac{i\partial}{\partial x}\right) u, Q_{t_0}^t \psi \right\rangle + \big\langle u, AQ_{t_0}^t \psi \big\rangle \\
&= -\left\langle \varphi\left(\frac{i\partial}{\partial x}\right) u, Q_{t_0}^t \psi \right\rangle + \left\langle \varphi\left(\frac{i\partial}{\partial x}\right) u, Q_{t_0}^t \psi \right\rangle = 0, \quad t \in [0, t_0).
\end{aligned}$$

Hence, $\big\langle u(t,\cdot), Q_{t_0}^t \psi \big\rangle$ is a constant. By using the properties of abstract functions, we deduce the relation

$$\lim_{t \to t_0} \big\langle u(t,\cdot), Q_{t_0}^t \psi \big\rangle = \langle u(t_0,\cdot), \psi \rangle = \text{const}$$

at any point $t_0 \in (0, +\infty)$. If $f = 0$ in (26), then

$$\mu \lim_{t \to +0} \langle u(t,\cdot), \psi \rangle - \sum_{k=1}^{m} \mu_k \langle u(t_k,\cdot), \psi \rangle = \mu c_0 - \sum_{k=1}^{m} \mu_k c_k = 0,$$

where $c_0, c_1, \ldots, c_m$ are arbitrary constants. This implies that $c_0 = c_1 = \ldots = c_m = 0$. Indeed, assume that this is not true, i.e., e.g., $c_0 \neq 0$. In this case, we get the relation

$$\mu - \sum_{k=1}^{m} \mu_k \alpha_k = 0,$$

where $\alpha_k = c_k/c_0$, $k \in \{1, \ldots, m\}$. Since $\mu, \mu_1, \ldots, \mu_m$ are fixed parameters and, in addition,

$$\mu > \sum_{k=1}^{m} \mu_k,$$

the obtained contradiction proves that $c_0 = 0$. Similarly, we can show that $c_1 = c_2 = \ldots = c_m = 0$. Thus, $\langle u(t_0, \cdot), \psi \rangle = 0$ for any $\psi \in S_1^{a_n}$, i.e., $u(t_0, x)$ is a zero functional from the space $\left(S_{1,*}^{a_n}\right)'$. Since $t_0 \in (0, +\infty)$ and t_0 is arbitrary, we conclude that $u(t, \cdot) = 0$ for all $t \in (0, +\infty)$.

Theorem 4 is proved.

Indeed, let

$$\varphi\left(\frac{i\partial}{\partial x}\right) = \left(I - \left(\frac{\partial}{\partial x}\right)^2\right)^{\omega/2}$$

be the operator of differentiation of fractional order in the space $S_1^{1/\omega} \equiv S_{k^k}^{n^{n/\omega}}$ constructed according to the function $\varphi(\sigma) = (1 + \sigma^2)^{\omega/2}$, $\sigma \in \mathbb{R}$, where $\omega \in [1, 2)$ is a fixed parameter, which is a multiplicator in the space $S_{1/\omega}^1 = S_{k^{k/\omega}}^{n^n}$. Then, by Theorem 4, the nonlocal multipoint (in time) problem for Eq. (12) with this operator is correctly solvable provided that the generalized function f in condition (26) is an element of the space $\left(S_{1,*}^{1/\omega}\right)'$.

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