ε-SHADING OPERATOR ON RIESZ SPACES AND ORDER CONTINUITY OF ORTHOGONALLY ADDITIVE OPERATORS

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ABSTRACT. Given a Riesz space E and $0 < e \in E$, we introduce and study an order continuous orthogonally additive operator which is an ε-approximation of the principal lateral band projection Q_e (the order discontinuous lattice homomorphism $Q_e \colon E \to E$ which assigns to any element $x \in E$ the maximal common fragment $Q_e(x)$ of e and x). This gives a tool for constructing an order continuous orthogonally additive operator with given properties. Using it, we provide the first example of an order discontinuous orthogonally additive operator which is both uniformly-to-order continuous and horizontally-to-order continuous. Another result gives sufficient conditions on Riesz spaces E and F under which such an example does not exist. Our next main result asserts that, if E has the principal projection property and F is a Dedekind complete Riesz space then every order continuous regular orthogonally additive operator $T \colon E \to F$ has order continuous modulus $|T|$. Finally, we provide an example showing that the latter theorem is not true for $E = C[0,1]$ and some Dedekind complete F. The above results answer two problems posed in a recent paper by O. Fotiy, I. Krasikova, M. Pliev and the second named author.

1. Introduction

We use standard terminology and notation on Riesz spaces as in [3]. In the next section, we provide with all necessary information on the lateral order and orthogonally additive operators (OAOs, in short) on Riesz spaces. In the present section, we describe our main results.

Basic order continuity properties of OAOs essentially differ from that of linear operators. Let E, F be Riesz spaces. Below we provide assertions for linear operators which and false for OAOs.

(1) If E has the principal projection property and F is Dedekind complete then every horizontally-to-order continuous linear operator $T \colon E \to F$ is order continuous [14, Proposition 3.9]. For OAOs this is false: if $0 \le p \le \infty$ then there exists a horizontally-to-order continuous orthogonally additive functional $f \colon L_p \to \mathbb{R}$ which is not order continuous (moreover, f is not uniformly-to-order continuous), see [5, Example 2.1].

(2) If F is Archimedean then every regular linear operator $T \colon E \to F$ is uniformly-to-order continuous [27, Proposition 4.6]. A typical example of a positive OAO which is not uniformly-to-order continuous is the principal lateral band projection Q_e (see the next section), see also (1).

(3) If F is Dedekind complete then an order bounded linear operator $T \colon E \to F$ is order continuous if and only if $|T|$ is. This is not true for OAOs: there is

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an order bounded orthogonally additive functional $f\colon \mathbb{R} \to \mathbb{R}$ such that $|f|$ is order continuous and f is not [5, Example 3.1].

Remark that the implication

$$(*) \qquad T \text{ is order continuous} \quad \Rightarrow \quad |T| \text{ is order continuous}$$

for OAOs is much more involved for investigation. It was one of the main questions considered in [5]. Some partial results were obtained there.

Proposition 1.1. [5, Proposition 3.2]. *Let E be a finite dimensional Archimedean Riesz space, F a Dedekind complete Riesz space and $T\colon E \to F$ an order continuous OAO. Then $|T|$ is order continuous.*

Theorem 1.2. [5, Theorem 3.8]. *Let E be a Riesz space with the principal projection property, Ω a nonempty set and $T\colon E \to \mathbb{R}^\Omega$ an order continuous OAO. Then the operator $|T|$ is order continuous as well.*

Theorem 1.3. [5, Corollary 3.5]. *Let E be a Riesz space with the principal projection property and F a Dedekind complete Riesz space. Then for every order continuous operator $T \in \mathcal{OA}_r(E, F)$ the following assertions are equivalent:*

1. *$|T|$ is order continuous;*
2. *T^+ is order continuous;*
3. *for every net (x_α) in E order convergent to some element $x \in E$, the following condition holds $\big(T^+(x_\alpha) - T^+(x)\big)^+ \xrightarrow{o} 0$.*

So the following natural questions remained unsolved.

Problem 1. [5]. *Under what assumptions on Riesz spaces E, F with F Dedekind complete every order bounded OAO $T\colon E \to F$ which is both horizontally-to-order continuous and uniformly-to-order continuous, is order continuous?*

Problem 2. [5]. *Do there exist a Riesz space with the principal projection property E, a Dedekind complete Riesz space F and an order continuous OAO $T\colon E \to F$ such that $|T|$ is not order continuous?*

The principal lateral band projection Q_e can serve as an "atomic" OAO in different constructions (as one of the summands) of an OAO with given properties. This technique was actively explored in [27] to prove the existence of OAOs with some pathological properties. However, Q_e is order discontinuous, which makes impossible construction of an order continuous OAO with given properties.

The first part of the present paper is devoted to construction of some order continuous operator Q_e^ε, which approximates Q_e as $\varepsilon \to 0+$.

2. Preliminary information

Let E be a Riesz space and $x, y \in E$. We say that x is a *fragment* of y (write $x \sqsubseteq y$) provided $x \perp y - x$. The set of all fragments of an element $e \in E$ is denoted by $\mathfrak{F}_e$. A disjoint sum in E is written using symbols $\bigsqcup, \sqcup$. So $y = \bigsqcup_{k=1}^m x_k$ means that $y = \sum_{k=1}^m x_k$ and $x_i \perp x_j$ as $i \neq j$. For instance, if $x \sqsubseteq y$ then $y = x \sqcup (y - x)$, and if $z = x \sqcup y$ then $x, y \in \mathfrak{F}_z$.

2.1. **The lateral order.** The relation $\sqsubseteq$ is a partial order on E, called the *lateral order* (see [16] for a systematic study of the lateral order). The supremum and infimum of a subset $A \subseteq E$ with respect to the lateral order (if exists) is denoted by $\bigcup A$ and $\bigcap A$ respectively (for a two-point set $A = \{x, y\}$ write $x \cup y$ and $x \cap y$). A subset $A \subseteq E$ is said to be laterally bounded if $A \subseteq \mathfrak{F}_e$ for some $e \in E$. Although any subset is laterally bounded from below by zero, a two-point set $\{x, y\} \subset E$ may not have a lateral infimum $x \cap y$ (which is the maximal common fragment of x and y), see [16, Example 3.11]. A Riesz space E is said to have the *intersection property* provided every two-point subset of E has a lateral infimum. The principal projection property implies the intersection property [16, Theorem 3.13], however the converse is not true ($C[0, 1]$ is a counterexample). A subset $G \subseteq E$ is called a *lateral ideal* provided $\mathfrak{F}_x \subseteq G$ for all $x \in G$, and $x \sqcup y \in G$ for all $x, y \in G$ with $x \perp y$. A lateral ideal G of E is said to be a lateral band if for every $A \subseteq G$ the existence of $\bigcup A$ implies $\bigcup A \in G$. Every order ideal is a lateral ideal and every band is a lateral band. For every $e \in E \setminus \{0\}$ the set $\mathfrak{F}_e$ is a lateral band which is not an order ideal. Moreover, $\mathfrak{F}_e$ is both the minimal lateral ideal and minimal lateral band containing e. We say that $\mathfrak{F}_e$ is the *principal lateral ideal* and *principal lateral band* generated by e. The notion of a lateral ideal (lateral band) is so important for the study of orthogonally additive operators (order continuous orthogonally additive operators) as well as the order ideals (respectively, bands) are important for the study of order bounded (respectively, order continuous) linear operators [16], [17].

Proposition 2.1 ([16], [27])**.** *Let E be a vector lattice and $e \in E$. Then the following assertions hold.*

1. *The set $\mathfrak{F}_e$ of all fragments of e is a Boolean algebra with zero 0, unit e with respect to the operations $\cup$ and $\cap$. Moreover, $x \cup y = (x_+ \vee y_+) - (x_- \vee y_-)$ and $x \cap y = (x_+ \wedge y_+) - (x_- \wedge y_-)$ for all $x, y \in \mathfrak{F}_e$.*
2. *Assume $e \geq 0$. Then the following holds.*
   - (a) *The lateral order $\sqsubseteq$ on $\mathfrak{F}_e$ coincides with the lattice order $\leq$.*
   - (b) *Let a nonempty subset A of $\mathfrak{F}_e$ have a lateral supremum $a = \bigcup A$ (respectively, a lateral infimum $a = \bigcap A$).*
     - (i) *If $y = \sup A$ (respectively, $y = \inf A$) exists in E then $y = a$.*
     - (ii) *If, moreover, E has the principal projection property then $\sup A$ (respectively, $\inf A$) exists in E and by* (i) *equals a.*

Remark that there exist a vector lattice E, an element $e \in E_+$ and subsets A and B of $\mathfrak{F}_e$ such that $\bigcup A$ and $\bigcap B$ exist, while $\sup A$ and $\inf B$ do not exist in E [27, Example 1.2].

2.2. **Orthogonally additive operators.** Let E be a Riesz space and X a real vector space. A function $T \colon E \to X$ is called an *orthogonally additive operator* (OAO in short) provided $T(x+y) = T(x)+T(y)$ for any disjoint elements $x, y \in E$. Obviously, if T is an OAO then $T(0) = 0$. The set of all OAOs from E to X is a real vector space with respect to the natural linear operations.

Let E, F be vector lattices. An OAO $T \colon E \to F$ is said to be:

- *positive* if $T(x) \geq 0$ for all $x \in E$;
- *regular* if T is a difference of two positive operators;
- *order bounded*, or an *abstract Uryson operator*, if it maps order bounded subsets of E to order bounded subsets of F;

- *laterally-to-order bounded* if the set $T(\mathfrak{F}_x)$ is order bounded in F for every $x \in E$;
- *disjointness preserving* if $Tx \perp Ty$ for every disjoint $x, y \in E$;
- *laterally non-expanding* if $T(x) \sqsubseteq x$ for all $x \in E$.

The positivity of OAOs is completely different from that of linear operators, and the only linear operator which is positive in the sense of OAOs is zero. A positive OAO need not be order bounded. Indeed, every function $T\colon \mathbb{R} \to \mathbb{R}$ with $T(0) = 0$ is an OAO, and, obviously, not all such functions are order bounded. Obviously, every laterally non-expanding OAO preserves disjointness. The kernel of a positive OAO is a lateral ideal [16, Proposition 6.4] and every lateral ideal is a kernel of some positive OAO [17, Theorem 3.1].

Denote the sets of all positive, regular, order bounded and laterally-to-order bounded OAOs from E to F by $\mathcal{OA}^+(E,F)$, $\mathcal{OA}_r(E,F)$, $\mathcal{U}(E,F)$ and $\mathcal{P}(E,F)$ respectively. Observe that $\mathcal{U}(E,F)$ is a vector subspace of $\mathcal{P}(E,F)$ and the inclusion $\mathcal{U}(E,F) \subset \mathcal{P}(E,F)$ is strict even for the one-dimensional case $E = F = \mathbb{R}$ ([26]). We endow $\mathcal{OA}_r(E,F)$ with the order $S \le T$ provided that $T - S$ is a positive OAO, that is, $Sx \le Tx$ for all $x \in E$. Then $\mathcal{OA}_r(E,F)$ becomes an ordered vector space.

The following theorem by Pliev and Ramdane generalizes a result by Mazón and Segura de León [14, Theorem 3.2.]

Theorem 2.2. [26, Theorem 3.6]. *Let E, F be Riesz spaces with F Dedekind complete. Then $\mathcal{OA}_r(E,F) = \mathcal{P}(E,F)$ and $\mathcal{OA}_r(E,F)$ is a Dedekind complete Riesz space. Moreover, for all $S, T \in \mathcal{OA}_r(E,F)$, $x \in E$ the following relations hold:*

(1) $(T \vee S)x = \sup\{Ty + Sz : x = y \sqcup z, \ y, z \in E\}$;
(2) $(T \wedge S)x = \inf\{Ty + Sz : x = y \sqcup z, \ y, z \in E\}$;
(3) $T^+x = \sup\{Ty : y \sqsubseteq x\}$;
(4) $T^-x = -\inf\{Ty : \ y \sqsubseteq x\}$;
(5) $|Tx| \le |T|x$.

Under the same assumptions on E and F, the set $\mathcal{U}(E,F)$ of all abstract Uryson operators is itself a Dedekind complete Riesz space possessing the same properties (1)-(5) [14, Theorem 3.2.]. Moreover, $\mathcal{U}(E,F)$ is an order ideal of $\mathcal{OA}_r(E,F)$ [26, Proposition 3.7], but not necessarily a band [26, Example 3.8].

2.3. **The principal lateral band projection Q_e.** A laterally non-expanding projection (that is, $T^2 = T$) is called a *lateral retraction*. A subset A of E is called a *lateral retract* if A is the image of some lateral retraction $T\colon E \to E$, that is, $T(E) = A$. A lateral band A of E, which is a lateral retract, is called a *projection lateral band*, and the lateral retraction of E onto A is called the *lateral band projection* of E onto A.

Theorem 2.3. [27, Theorem 1.6]. *Let E be a vector lattice with the intersection property. Then for every $e \in E \setminus \{0\}$ the function $Q_e\colon E \to E$ defined by setting*

$$Q_e(x) = x \Cap e \quad \text{for all } x \in E$$

is the lateral band projection of E onto $\mathfrak{F}_e$.

In Theorem 3.2 we give explicit formula (3.4) for $x \Cap e$ in terms of lattice operations over x and e if $x - e$ is a projective element of E.

2.4. **The intersection property is a lateral analogue of the principal projection property.** Recall that $y \in E$ is called a *projection element* of E provided $E = E_y \oplus E_y^{dd}$, where by E_y we denote the minimal order ideal containing y. In this case the order projection P_y of E onto E_y is given (see [3, Theorem 1.47]) by

$$P_y\, x = \bigvee_{n=1}^{\infty} (x \wedge n|y|), \ \ x \in E^+. \tag{2.1}$$

We need the following property of P_y. By (3) of [3, Theorem 1.44],

$$(\forall u, v \in E) \ \ P_y\, u \perp (v - P_y\, v). \tag{2.2}$$

A Riesz space E is said to have the *principal projection property* provided every element of E is a projection element.

Let E be a vector lattice and $x, y \in E$. We say that x is *laterally disjoint* to y and write $x \dagger y$ if $\mathfrak{F}_x \cap \mathfrak{F}_y = \{0\}$. Two subsets A and B of E are said to be *laterally disjoint* (write $A \dagger B$) if $x \dagger y$ for every $x \in A$ and $y \in B$. The *laterally disjoint complement* to a subset A of E is defined as follows: $A^\dagger := \{x \in E : (\forall a \in A)\, x \dagger a\}$. Note that $x \perp y$ implies $x \dagger y$ for all $x, y \in E$ and the converse is false. However, $x \dagger y$ implies $x \perp y$ for every laterally bounded pair $x, y \in E$. An element e of a vector lattice E is called a *laterally projection element* provided E is decomposed into a nonlinear direct sum $E = \mathfrak{F}_e \sqcup \mathfrak{F}_e^\dagger$, that is, every $x \in E$ has a unique representation

$$x = y \sqcup z, \ \ \text{where} \ \ y \in \mathfrak{F}_e \ \ \text{and} \ \ z \in \mathfrak{F}_e^\dagger. \tag{2.3}$$

Proposition 2.4. [25, Proposition 4.9]. *A vector lattice E has the intersection property if and only if every element of E is laterally projective. Moreover, representation* (2.3) *of any $x \in E$ is given by $x = Q_e x \sqcup (x - Q_e x)$, where Q_e is the principal lateral band projection.*

2.5. **Different types of order convergence and order continuity.** A net $(x_\alpha)_{\alpha \in A}$ in a Riesz space E converges to a limit $x \in E$:

- *strongly order* if there is a net $(u_\alpha)_{\alpha \in A}$ in E such that $u_\alpha \downarrow 0$ and $|x_\alpha - x| \leq u_\alpha$ for some $\alpha_0 \in A$ and all $\alpha \geq \alpha_0$ (write $x_\alpha \xrightarrow{s.o} x$);
- *weakly order* if there is a net $(v_\beta)_{\beta \in B}$ in E such that $v_\beta \downarrow 0$ and for every $\beta \in B$ there exists $\alpha_0 \in A$ such that $|x_\alpha - x| \leq v_\beta$ for all $\alpha \geq \alpha_0$ (write $x_\alpha \xrightarrow{w.o} x$);
- *horizontally*[1] if $x_\alpha \sqsubseteq x_\beta$ for all $\alpha < \beta$ and $\bigsqcup_{\alpha \in A} x_\alpha = x$ (write $x_\alpha \xrightarrow{\mathrm{h}} x$);
- *e-uniformly*, where $e \in E^+$ if
$$\forall n \in \mathbb{N} \ \ \exists \alpha_0 \in A \ \ \forall \alpha \geq \alpha_0 \ \ \ |x_\alpha - x| \leq \frac{1}{n} e;$$
in this case we write $x_\alpha \overset{e}{\rightrightarrows} x$;
- *uniformly*, provided $(x_\alpha)_{\alpha \in A}$ converges to x e-uniformly for some $e \in E^+$; in this case we write $x_\alpha \rightrightarrows x$.

Every strongly order convergent net weakly converges to the same limit, but the converse is false [2]. However, the strong and weak order convergence are equivalent if either E is Dedekind complete or the net $(x_\alpha)_{\alpha \in A}$ is monotone. In these two cases we write $x_\alpha \xrightarrow{o} x$. Note that one can equivalently replace the condition $\bigsqcup_{\alpha \in A} x_\alpha = x$ with $x_\alpha \xrightarrow{o} x$ in the definition of horizontal convergence

[1] *laterally* or *up-laterally* in other terminology

under the assumption $x_\alpha \sqsubseteq x_\beta$ for all $\alpha < \beta$. The uniform convergence of a net implies the strong order convergence the net to the same limit.

The order continuity of operators we understand in the sense of strong order convergence. More precisely, let E, F be Riesz spaces. An OAO $T\colon E \to F$ is said to be:

- *order continuous* provided for any $x \in E$ and any net $(x_\alpha)_{\alpha \in A}$ in E the condition $x_\alpha \xrightarrow{s.o} x$ implies $T(x_\alpha) \xrightarrow{s.o} T(x)$;
- *horizontally-to-order continuous* provided for any $x \in E$ and any net $(x_\alpha)_{\alpha \in A}$ in E the condition $x_\alpha \xrightarrow{h} x$ implies $T(x_\alpha) \xrightarrow{s.o} T(x)$;
- *uniformly-to-order continuous* provided for any $x \in E$ and any net $(x_\alpha)_{\alpha \in A}$ in E the condition $x_\alpha \rightrightarrows x$ implies $T(x_\alpha) \xrightarrow{s.o} T(x)$.

3. ε-shading operator

The principal lateral band projection Q_e (see Theorem 2.3) gives an important tool for constructing examples of OAOs defined on a Riesz space with the intersection property. However, Q_e is not order continuous (because $Q_e\big((1 - \frac{1}{n})\, e\big) = 0 \neq e = Q_e(e)$ for all $n \in \mathbb{N}$, however $(1 - \frac{1}{n})\, e \xrightarrow{s.o} e$). In the present section, we construct a "blurring" version of Q_e, which mainly has similar properties and is order continuous. Another superiority of this version is that it acts in an arbitrary Riesz space.

Let E be a Riesz space, $0 < e \in E$ and $0 < \varepsilon < 1$. Define a map $Q_e^\varepsilon\colon E \to E$ by setting

$$Q_e^\varepsilon(x) = \frac{1}{\varepsilon}\Big(\big(x - (1-\varepsilon)\, e\big)^+ \wedge \big((1+\varepsilon)\, e - x\big)^+\Big), \ \ x \in E. \tag{3.1}$$

The map Q_e^ε defined by (3.1) will be called the *ε-shading operator generated by e*.

Lemma 3.1. *Let E be a Riesz space, $0 < e \in E$ and $0 < \varepsilon < 1$. Then for every $x \in E$ one has*

$$Q_e^\varepsilon(x) = \left(e - \frac{1}{\varepsilon}|x - e|\right)^+ = e - e \wedge \frac{1}{\varepsilon}|x - e|. \tag{3.2}$$

Proof. Remark that, by the well known formula [3, Theorem 1.7]

$$(\forall s, t \in E) \quad s = (s-t)^+ + s \wedge t, \tag{3.3}$$

it is enough to prove one of the equalities. For every $x \in E$, using theorems 1.3 and 1.8 of [3], we obtain

$$\begin{aligned}
\varepsilon e - \varepsilon Q_e^\varepsilon(x) &\overset{(3.3)}{=} \varepsilon e - \big(x - x \wedge (1-\varepsilon)\, e\big) \wedge \big((1+\varepsilon)\, e - (1+\varepsilon)\, e \wedge x\big) \\
&= \varepsilon e + \big(x \wedge (1-\varepsilon) e - x\big) \vee \big((1+\varepsilon) e \wedge x - (1+\varepsilon) e\big) \\
&= \big(x \wedge (1-\varepsilon) e + \varepsilon e - x\big) \vee \big((1+\varepsilon) e \wedge x - e\big) \\
&= \big(\varepsilon e \wedge (e - x)\big) \vee \big(\varepsilon e \wedge (x - e)\big) = \varepsilon e \wedge |x - e|.
\end{aligned}$$

□

The following theorem collects main properties of the ε-shading operator.

Theorem 3.2. *Let E be a Riesz space, $0 < e \in E$ and $0 < \varepsilon < 1$. Then the map Q_e^ε defined by* (3.1) *is a positive disjointness preserving order continuous OAO possessing the following properties.*

(i) $(\forall x \in E)\ 0 \le Q_e^\varepsilon(x) = Q_e^\varepsilon(x^+) \le x^+ \wedge e$.
(ii) $(\forall x, y \in E)\ |Q_e^\varepsilon(x) - Q_e^\varepsilon(y)| \le \frac{2}{\varepsilon}\,|x-y|$.
(iii) $(\forall t \in \mathfrak{F}_e)\ Q_e^\varepsilon(t) = t$;
(iv) *For every $x \in E$ such that either $x \le (1-\varepsilon)\,e$ or $x \ge (1+\varepsilon)\,e$ one has $Q_e^\varepsilon(x) = Q_e(x) = 0$.*
(v) *For every $x \in E$ if $0 < \varepsilon' < \varepsilon'' < 1$ then $Q_e^{\varepsilon'}(x) \le Q_e^{\varepsilon''}(x)$.*
(vi) *If $x - e$ is a projection element of E then both $\bigwedge_{n=1}^\infty Q_e^{1/n}(x)$ and $x \cap e$ exist and*

$$\bigwedge_{n=1}^{\infty} Q_e^{1/n}(x) = e - P_{x-e}\, e = x \cap e = x - P_{x-e}\, x. \tag{3.4}$$

(vii) *Let $e = e' \sqcup e''$. Then $Q_{e'}^\varepsilon(x) \sqcup Q_{e''}^\varepsilon(x) = Q_e^\varepsilon(x)$ for all $x \in E$. If, moreover, e' and e'' are projection elements of E then $Q_{e'}^\varepsilon \sqcup Q_{e''}^\varepsilon = Q_e^\varepsilon$.*

For the proof, we need the following lemma.

Lemma 3.3. *Let E be a Riesz space and $u, v \in E^+$. Then the map $Q\colon E \to E$ defined by setting*

$$Q(x) = (x-u)^+ \wedge (v-x)^+ \quad \textit{for all} \quad x \in E$$

is a positive disjointness preserving order continuous OAO possessing the following properties:

(i) $(\forall x \in E)\ 0 \le Q(x) = Q(x^+) \le x^+$;
(ii) $(\forall x, y \in E)\ |Q(x) - Q(y)| \le 2|x-y|$.

Proof. First fix any $x, y \in E^+$ with $x \perp y$ and prove

$$Q(x+y) = Q(x) + Q(y). \tag{3.5}$$

Now we show that

$$(v-y) \wedge x = v \wedge x \quad \text{and} \quad (v-x) \wedge y = v \wedge y. \tag{3.6}$$

Indeed, on the one hand,

$$|v \wedge x - (v-y) \wedge x| \le |v - (v-y)| = y.$$

On the other hand,

$$|v \wedge x - (v-y) \wedge x| \le |v \wedge x| + |(v-y) \wedge x| \le 2x.$$

Hence, $|v \wedge x - (v-y) \wedge x| \le y \wedge (2x) = 0$ and the first equality in (3.6) is proved. The second one is similar. Now (3.6) implies

$$(v-y)^+ \wedge x = v \wedge x \quad \text{and} \quad (v-x)^+ \wedge y = v \wedge y. \tag{3.7}$$

Indeed,

$$(v-y)^+ \wedge x = \big((v-y) \vee 0\big) \wedge x = \big((v-y) \wedge x\big) \vee 0 \overset{(3.6)}{=} v \wedge x.$$

Taking into account that $x+y=x\vee y$, we obtain

$$\begin{aligned} Q(x+y) &= (x\vee y-u)^+\wedge(v-x\vee y)^+ \\ &= \big((x-u)\vee(y-u)\big)^+\wedge\big((v-x)\wedge(v-y)\big)^+ \\ &= \big((x-u)^+\vee(y-u)^+\big)\wedge\big((v-x)^+\wedge(v-y)^+\big) \\ &= w_1\vee w_2, \end{aligned} \tag{3.8}$$

where

$$w_1=(x-u)^+\wedge(v-x)^+\wedge(v-y)^+ \quad\text{and}\quad w_2=(y-u)^+\wedge(v-x)^+\wedge(v-y)^+.$$

Observe that $w_1\le x$ and $w_1\le v$. Hence

$$w_1=w_1\wedge x \overset{(3.7)}{=} (x-u)^+\wedge(v-x)^+\wedge v\wedge x=(x-u)^+\wedge(v-x)^+=Q(x).$$

Analogously, $w_2\le y$, $w_2\le v$ and hence

$$w_2=w_2\wedge y \overset{(3.7)}{=} (y-u)^+\wedge v\wedge y\wedge(v-y)^+=(y-u)^+\wedge(v-y)^+=Q(y).$$

Taking into account that $0\le w_1\le x$ and $0\le w_2\le y$, we obtain that $w_1\perp w_2$ and hence, $w_1\vee w_2=w_1+w_2=Q(x)+Q(y)$. By (3.8), one gets (3.5).

To prove (3.5) for the general case of $x,y\in E$ with $x\perp y$, by the above, it is enough to prove that

$$Q(x)=Q(x^+) \quad\text{for all}\quad x\in E. \tag{3.9}$$

Fix any $x\in E$. We need two claims and the following known elementary fact (see item (2) of [3, Theorem 1.7]): $\forall r,s,t\in E$

$$|s\vee r-t\vee r|\le|s-t| \quad\text{and}\quad |s\wedge r-t\wedge r|\le|s-t|. \tag{3.10}$$

Claim 1. $(x-u)^+=(x^+-u)^+$.

Proof of Claim 1. Observe that

$$(\forall z\in E)(\forall e\in E^+)\quad z\wedge e=z^+\wedge e-z^-. \tag{3.11}$$

Indeed, $(z\wedge e)^+=(z\wedge e)\vee 0=(z\vee 0)\wedge(e\vee 0)=z^+\wedge e$ and $(z\wedge e)^-=\big(-(z\wedge e)\big)\vee 0=(-z)\vee(-e)\vee 0=z^-$, which implies (3.11). By (3.11), for every $e\in E^+$ one has

$$\begin{aligned} (x+y)\wedge e &= (x+y)^+\wedge e-(x+y)^- \\ &= (x^++y^+)\wedge e-(x^-+y^-) \\ &= x^+\wedge e+y^+\wedge e-x^--y^- \\ &= x\wedge e+y\wedge e. \end{aligned} \tag{3.12}$$

Now we obtain

$$x\wedge u=(x^+-x^-)\wedge u \overset{(3.12)}{=} x^+\wedge u+(-x^-)\wedge u=x^+\wedge u-x^-$$

and hence

$$(x-u)^+ \overset{(3.3)}{=} x-x\wedge u=x-x^+\wedge u+x^-=x^+-x^+\wedge u \overset{(3.3)}{=} (x^+-u)^+.$$

□

Claim 2. $(v-x)^+\wedge x^+=(v-x^+)^+\wedge x^+$.

Proof of Claim 2. We have

$$\begin{aligned} 0 \le w &:= (v-x)^+ \wedge x^+ - (v-x^+)^+ \wedge x^+ \\ &= (v - x^+ + x^-)^+ \wedge x^+ - (v-x^+)^+ \wedge x^+ \\ &\overset{(3.10)}{\le} (v - x^+ + x^-)^+ - (v-x^+)^+ \\ &= (v - x^+ + x^-) \vee 0 - (v-x^+) \vee 0 \overset{(3.10)}{\le} x^-. \end{aligned} \tag{3.13}$$

On the other hand, since $(v-x)^+ \wedge x^+ \perp x^-$ and $(v-x^+)^+ \wedge x^+ \perp x^-$, we have $w \perp x^-$. Together with (3.13), this yields that $w = 0$. □

Now we continue the proof of Lemma 3.3. By (3.3) and Claim 1,

$$Q(x) = (x^+ - x^+ \wedge u) \wedge (v-x)^+ \le x^+ \wedge (v-x)^+ \le x^+ \tag{3.14}$$

and analogously, $Q(x^+) \le x^+$. Hence,

$$Q(x) = Q(x) \wedge x^+ \quad \text{and} \quad Q(x^+) = Q(x^+) \wedge x^+.$$

Thus, by Claim 2,

$$\begin{aligned} Q(x) &= (x^+ - u)^+ \wedge (v-x)^+ \wedge x^+ \\ &= (x^+ - u)^+ \wedge (v - x^+)^+ \wedge x^+ = Q(x^+) \end{aligned}$$

and (3.9) is proved. So, Q is an OAO.

Item (i) is already proved by (3.9) and (3.14). Q preserves disjointness by (i). The order continuity of Q follows from (ii). So, it remains to prove (ii). Observe that, for every $a, b, c, d \in E$ one has

$$|a \wedge b - c \wedge d| \le |a \wedge b - a \wedge d| + |a \wedge d - c \wedge d| \overset{(3.10)}{\le} |b-d| + |a-c|.$$

Hence, for every $x, y \in E$ we obtain

$$|Q(x) - Q(y)| \le \big|(x-u)^+ - (y-u)^+\big| + \big|(v-x)^+ - (v-y)^+\big| \overset{(3.10)}{\le} 2|x-y|.$$

□

Proof of Theorem 3.2. By Lemma 3.3, Q_e^ε is a positive order continuous OAO and (ii) holds true.

(i) The part $(\forall x \in E)\ 0 \le Q_e^\varepsilon(x) = Q_e^\varepsilon(x^+) \le x^+$ follows from Lemma 3.3, and the inequality $Q_e^\varepsilon(x^+) \le e$ follows from Lemma 3.1.

(iii) Fix any $t \in \mathfrak{F}_e$. By (3.1) and (3.3),

$$Q_e^\varepsilon(t) = \frac{1}{\varepsilon}\Big(\big(t - t \wedge (1-\varepsilon)\, e\big) \wedge \big((1+\varepsilon)\, e - t \wedge (1+\varepsilon)\, e\big)\Big). \tag{3.15}$$

Since $(1-\varepsilon)\, t \le t$, $\ t \le e$ and $e = t \sqcup (e-t) = t \vee (e-t)$, one has

$$(1-\varepsilon)\, t \le t \wedge (1-\varepsilon)\, e = t \wedge \big((1-\varepsilon)\, t \vee (1-\varepsilon)(e-t)\big) = (1-\varepsilon)\, t \vee 0 = (1-\varepsilon)\, t$$

and hence

$$t \wedge (1-\varepsilon)\, e = (1-\varepsilon)\, t. \tag{3.16}$$

On the other hand, $(1+\varepsilon)\, t \le (1+\varepsilon)\, e$ implies

$$\varepsilon t \wedge \big((1+\varepsilon)\, e - t\big) = \varepsilon t. \tag{3.17}$$

Finally, (3.15), (3.16) and (3.17) together give

$$Q_e^\varepsilon(t) = \frac{1}{\varepsilon}\Big(\big(t - t + \varepsilon t\big) \wedge \big((1+\varepsilon)\, e - t\big)\Big) = t.$$

(iv) follows from (3.1).

(v) follows from Lemma 3.1.

(vi) Assume $x - e$ is a projection element. Then by (2.1) and Lemma 3.1

$$z := e - P_{x-e}\, e = e - \bigvee_{n=1}^{\infty} (e \wedge n|x-e|) = \bigwedge_{n=1}^{\infty} (e - e \wedge n|x-e|) = \bigwedge_{n=1}^{\infty} Q_e^{1/n}(x). \tag{3.18}$$

Show that $z = e \Cap x$. By (2.2), $e = P_{x-e}\, e \sqcup z$, which implies that $z \sqsubseteq e$. Then

$$\begin{aligned} z \wedge |x - z| &= (e - P_{x-e}\, e) \wedge \big|x - e + P_{x-e}\, e\big| \\ &= (e - P_{x-e}\, e) \wedge \big|P_{x-e}(x - e) + P_{x-e}\, e\big| \\ &= (e - P_{x-e}\, e) \wedge P_{x-e}\, x \overset{(2.2)}{=} 0, \end{aligned}$$

which yields $z \sqsubseteq x$. Assume $t \in F_x \cap F_e$ and prove that $t \sqsubseteq z$. Observe that for every $n \in \mathbb{N}$ the relation $t \sqsubseteq e$ implies $t = Q_e^{1/n}(t)$ by (iii), and the relation $t \sqsubseteq x$ implies $Q_e^{1/n}(t) \le Q_e^{1/n}(x)$ by the positivity of $Q_e^{1/n}$. Hence, $t \le \bigwedge_{n=1}^{\infty} Q_e^{1/n}(x) \overset{(3.18)}{=} z$. By (2a) of Proposition 2.1, the orders $\sqsubseteq$ and $\le$ coincide on $\mathfrak{F}_e$ and therefore $t \sqsubseteq z$. The relation $z = e \Cap x$ is proved. Together with (3.18) this gives (vi).

(vii) Given any $x \in E$, one has

$$w := \big|\varepsilon e' \wedge (x - e) - \varepsilon e' \wedge (x - e')\big| \le x - e - x + e' = e''.$$

On the other hand, $0 \le w \le \varepsilon e'$. Hence, $0 \le w \le e' \wedge e'' = 0$, which implies $w = 0$. Analogously, $\varepsilon e' \wedge (e - x) = \varepsilon e' \wedge (e' - x)$. By that

$$\begin{aligned} \varepsilon e' \wedge |x - e| &= \varepsilon e' \wedge \big((x - e) \vee (e - x)\big) \\ &= \big(\varepsilon e' \wedge (x - e)\big) \vee \big(\varepsilon e' \wedge (e - x)\big) \\ &= \big(\varepsilon e' \wedge (x - e')\big) \vee \big(\varepsilon e' \wedge (e' - x)\big) \\ &= \varepsilon e' \wedge |x - e'| = \varepsilon e' - \varepsilon Q_{e'}^\varepsilon(x). \end{aligned}$$

Analogously, $e'' \wedge \frac{1}{\varepsilon}|x - e| = e'' - Q_{e''}^\varepsilon(x)$. Hence

$$\begin{aligned} Q_{e'}^\varepsilon(x) + Q_{e''}^\varepsilon(x) &= e' - e' \wedge \frac{1}{\varepsilon}|x - e| + e'' - e'' \wedge \frac{1}{\varepsilon}|x - e| \\ &= e - \Big(\big(e' \wedge \frac{1}{\varepsilon}|x - e|\big) \vee \big(e'' \wedge \frac{1}{\varepsilon}|x - e|\big)\Big) \\ &= e - (e' \vee e'') \wedge \frac{1}{\varepsilon}|x - e| = Q_e^\varepsilon(x) \end{aligned}$$

and the first part of (vii) is proved. Assume now that e' and e'' are projection elements. By (2) of Theorem 2.2,

$$\big(Q_{e'}^\varepsilon \wedge Q_{e''}^\varepsilon\big)(x) = \inf\Big\{Q_{e'}^\varepsilon(y) + Q_{e''}^\varepsilon(z) :\ x = y \sqcup z\Big\}. \tag{3.19}$$

Set $y = P_{e''}x \sqcup (x - P_e x)$ and $z = P_{e'}x$. Then

$$\begin{aligned} Q_{e'}^\varepsilon(y) &= Q_{e'}^\varepsilon\big(P_{e''}x\big) + Q_{e'}^\varepsilon\big(x - P_e x\big) \\ &\overset{\text{(i)}}{\le} \big(P_{e''}x\big) \wedge e' + \big(x - P_e x\big) \wedge e' \overset{(2.2)}{=} 0. \end{aligned}$$

Analogously, $Q_{e''}^\varepsilon(z) = 0$ which confirms by (3.19) that $Q_{e'}^\varepsilon \perp Q_{e''}^\varepsilon$. ∎

4. A uniformly-to-order continuous and horizontally-to-order continuous OAO, which is not order continuous

In the present section, using the technique of ε-shading operators, we provide the first example of such an operator. Moreover, it is a functional, that is, with values in $\mathbb{R}$. It is defined on $C[0,1]$ possessing the intersection property, but failing to have the principal projection property.

Theorem 4.1. *There exists an order discontinuous orthogonally additive functional $f\colon C[0,1] \to [0,1]$, which is uniformly-to-order continuous and horizontally-to-order continuous.*

Proof. Define a sequence $(e_n)_{n=1}^{\infty}$ in $C[0,1]$ as follows. Let $e_n\colon [0,1] \to [0,1]$ be the piece-wise linear function with nodes at the points $\big(0, \frac{1}{2} + \frac{1}{n}\big)$, $\big(\frac{1}{n}, \frac{1}{n}\big)$ and $\big(1, \frac{1}{n}\big)$ of $\mathbb{R}^2$, that is,

$$e_n(t) = \begin{cases} \frac{1}{2} + \frac{1}{n} - \frac{nt}{2}, & \text{if } t \in \big[0, \frac{1}{n}\big) \\ \frac{1}{n}, & \text{if } t \in \big[\frac{1}{n}, 1\big] \end{cases}, \quad n \in \mathbb{N}.$$

Observe that $e_n \in C[0,1]$ and $e_{n+1}(t) < e_n(t)$ for all $t \in [0,1]$ and $n \in \mathbb{N}$. Choose numbers $\varepsilon_n > 0$ so that

$$(\forall n \in \mathbb{N})(\forall t \in [0,1]) \ \ e_{n+1}(t)(1 + \varepsilon_{n+1}) < e_n(t)(1 - \varepsilon_n). \tag{4.1}$$

Then define a functional $f\colon C[0,1] \to [0,1]$ by setting

$$f(x) = \sum_{n=1}^{\infty} n \min_{t \in [0,1]} \big(Q_{e_n}^{\varepsilon_n}(x)\big)(t), \tag{4.2}$$

where $Q_{e_n}^{\varepsilon_n}$ is the ε_n-shading operator generated by e_n. To show that the functional is well defined by (4.2), set

$$B_n := \big\{x \in C[0,1] : \ (\forall t \in [0,1]) \ e_n(t)(1 - \varepsilon_n) < x(t) < e_n(t)(1 + \varepsilon_n)\big\}, \ n \in \mathbb{N}.$$

By (4.1), $(B_n)_{n=1}^{\infty}$ is a disjoint sequence of open subsets of $C[0,1]$. By (3.1),

$$(\forall x \in C[0,1])(\forall n \in \mathbb{N}) \ \ \min_{t \in [0,1]} \big(Q_{e_n}^{\varepsilon_n}(x)\big)(t) > 0 \ \ \Rightarrow \ \ x \in B_n. \tag{4.3}$$

By (4.1), this implies that for every $x \in C[0,1]$ at most one of the summands in (4.2) is nonzero, and so f is well defined by (4.2).

Prove that f is orthogonally additive. Let $x, y \in C[0,1]$ and $x \perp y$. If either $x = 0$ or $y = 0$ then $f(x+y) = f(x) + f(y)$, because $f(0) = 0$. Now suppose $x \neq 0 \neq y$. Then $x \perp y$ implies that there is $t \in [0,1]$ such that $x(t) = y(t) = 0$. Taking into account that every element of B_n takes nonzero values only for all $n \in \mathbb{N}$, this yields by (4.3) that $f(x) = f(y) = f(x+y) = 0$. So f is orthogonally additive.

Prove that f is uniformly-to-order continuous. Since the uniformly convergence in $C[0,1]$ is equivalent to the norm-convergence, it is enough to show that f is norm-to-order continuous. According to Theorem 3.2 (ii), every function

$$f_n(x) = n \min_{t \in [0,1]} \big(Q_{e_n}^{\varepsilon_n}(x)\big)(t)$$

is norm-to-order continuous. By (4.3),

$$\operatorname{supp} f_n = \{x \in C[0,1] : f_n(x) \neq 0\} = B_n.$$

Since $(e_n)_{n=1}^\infty$ has no a norm-convergent subsequence and $\varepsilon_n \to 0$, the sequence $(B_n)_{n=1}^\infty$ is locally finite with respect to the norm. Therefore, f is norm-to-order continuous as locally finite sum of norm-to-order continuous functions.

Prove that f is horizontally-to-order continuous. Let $x \in C[0,1]$ and $(x_\alpha)_{\alpha\in A}$ be a net in $C[0,1]$ such that $x_\alpha \xrightarrow{h} x$. Consider two cases.

(i) $(\exists \alpha_0 \in A)(\forall \alpha \geq \alpha_0)\ x_\alpha = x$. Then obviously $f(x_\alpha) \to f(x)$.

(ii) $(\forall \beta \in A)(\exists \alpha \geq \beta)\ x_\alpha \neq x$. Since $x_\alpha \sqsubseteq x_\beta \sqsubseteq x$ for all $\alpha < \beta$, we have that $(\forall \alpha \in A)\ x_\alpha \neq x$. By the peculiarity of $C[0,1]$, for every x_α vanishes at some point of $[0,1]$, as well as x. By (4.3), $f(x) = 0 = f(x_\alpha)$ for all $\alpha \in A$.

The horizontal-to-order continuity of f is proved.

To prove that f is not order continuous, observe that $e_n \xrightarrow{s.o} 0$ and $f(e_n) = 1 \neq 0 = f(0)$ for all $n \in \mathbb{N}$. □

5. Continuity of horizontally and uniformly continuous operators

In this section, we provide sufficient conditions on Riesz spaces E, F, under which every horizontally-to-order continuous and uniformly-to-order continuous OAO is order continuous, giving a partial answer to Problem 1.

Let F be a Riesz space. By $\mathcal{D}(F)$ we denote the set of all Riesz spaces E such that every abstract Uryson operator $T\colon E \to F$ which is both horizontally-to-order continuous and uniformly-to-order continuous, is order continuous. Next is our main result of the section.

Our first result here is the following theorem.

Theorem 5.1. *Let E be a Riesz space such that for every net $(u_\alpha)_{\alpha\in A}$ in E with $u_\alpha \downarrow 0$ there exists a net $(E_i)_{i\in I}$ of projection bands E_i in E such that*

(1) *$P_i(u_\alpha) \rightrightarrows_\alpha 0$ for every $i \in I$, where P_i is the order projection of E onto E_i;*
(2) *$E_i \subseteq E_j$ for every $i, j \in I$ with $i < j$;*
(3) *$P_i(x) \xrightarrow{h}_i x$ for every $x \in E$.*

Then $E \in \mathcal{D}(F)$ for every Riesz space F with the principal projection property.

For the proof of Theorem 5.1 we need some lemmas.

Lemma 5.2. *Let E be a Dedekind complete Riesz space and A an upper bounded subset of E with $0 \in A$. If $v \in E^+$ satisfies*

$$\sup\{y \in A : x + y \in A\} \geq v$$

for every $x \in A$ then $v = 0$.

Proof. For every $x \in A$ we set $A_x = \{y \in A : x + y \in A\}$ and $u_x = \sup A_x$. Assume that $v > 0$. Since E is an Archimedean Riesz space, there exists a number $n \in \mathbb{N}$ such that $\frac{n}{2}v \not\leq u_0 = \sup A$. Then $w_0 := (\frac{n}{2}v - u_0)^+ > 0$. Let $x_0 = 0$. Now we construct finite sequences $(x_k)_{k=1}^n$ and $(w_k)_{k=1}^n$ of elements $x_k \in A$ and $w_k \in E^+$ such that for every $k = 1, \dots, n$

(1) $x_0 + \dots + x_{k-1} \in A$;
(2) $x_k \in A_{x_0+\dots+x_{k-1}}$;
(3) $w_k \in B_{w_{k-1}}$ where $B_{w_{k-1}}$ is the principal band generated by w_{k-1};
(4) $P_k(x_k - \frac{1}{2}v) = w_k > 0$, where P_k is the order projection onto B_{w_k}.

Since $w_0 \le \frac{n}{2}v$, $w_0 \in B_v$ where B_v is the principal band generated by v. Therefore, $P_0(v) > 0$ where P_0 is the order projection onto B_{w_0}. Since $u_0 \ge v$ and

$$P_0(u_0) = P_0(\sup A_0) = \sup\{P_0(x) : x \in A_0\},$$

one has

$$P_0(u_0) \ge P_0(v) > P_0(\tfrac{1}{2}v)$$

and there exists $x_1 \in A_0$ such that $P_0(x_1) \not\le P_0(\frac{1}{2}v)$. We set

$$w_1 = (P_0(x_1) - P_0(\tfrac{1}{2}v))^+.$$

Then $w_1 \in B_{w_0}$ and $w_1 > 0$. Moreover,

$$P_1(x_1 - \tfrac{1}{2}v) = P_1(P_0(x_1 - \tfrac{1}{2}v)) = P_1(w_1) = w_1.$$

Since $w_1 \in B_{w_0}$ and $w_0 \in B_v$, one has $w_1 \in B_v$ and $P_1(v) > 0$. Hence, taking into account that (by the choice of x_1) $x_0 + x_1 \in A$, we obtain

$$P_1(u_{x_1}) = \sup\{P_1(x) : x \in A_{x_0+x_1}\} \ge P_1(v) > P_1(\tfrac{1}{2}v).$$

Therefore, there exists $x_2 \in A_{x_0+x_1}$ such that $P_1(x_2) \not\le P_1(\frac{1}{2}v)$. We set

$$w_2 = (P_1(x_2) - P_1(\tfrac{1}{2}v))^+.$$

It is clear that $w_2 \in B_{w_1}$ and $w_2 > 0$. Moreover,

$$P_2(x_2 - \tfrac{1}{2}v) = P_2(P_1(x_2 - \tfrac{1}{2}v)) = P_2(w_2) = w_2.$$

To complete the construction of $(x_k)_{k=1}^n$ and $(w_k)_{k=1}^n$, it remains to repeat the reasoning $n-2$ times.

Now we consider the element

$$x = x_1 + x_2 + \dots x_n.$$

By (2), $x \in A$. On the one hand

$$P_n(\tfrac{n}{2}v - x) \ge P_n(\tfrac{n}{2}v - u_0) = P_n(P_0(\tfrac{n}{2}v - u_0)) = P_n(w_0) \ge 0.$$

But on the other hand we have that

$$\begin{aligned} P_n(x - \tfrac{n}{2}v) &= \sum_{k=1}^n P_n(x_k - \tfrac{1}{2}v) \\ &= \sum_{k=1}^n P_n(P_k(x_k - \tfrac{1}{2}v)) \\ &= \sum_{k=1}^n P_n(w_k) \ge w_n > 0, \end{aligned}$$

a contradiction. □

Lemma 5.3. *Let E be a Riesz space and F a Dedekind complete Riesz space, $T\colon E \to F$ a (bounded???) function and $x_0 \in E$. Then the following conditions are equivalent:*

(i) *T is order continuous at x_0;*

(ii) *for every net $(u_\alpha)_{\alpha\in A}$ in E such that $u_\alpha \downarrow 0$ we have that*

$$\inf_{\alpha\in A} v_\alpha = 0 = \inf_{\alpha\in A} w_\alpha, \tag{5.1}$$

where $v_\alpha = \sup\{T(x) - T(x_0) : |x - x_0| \le u_\alpha\}$ and $w_\alpha = -\inf\{T(x) - T(x_0) : |x - x_0| \le u_\alpha\}$.

Proof. (i) ⇒ (ii). It is enough to prove the first equality only, because the second equality coincide with the first one for $-T$ in place of T. If A has a maximal element then the claim of the lemma is obvious. Assume A has no maximal element. Fix any net $(u_\alpha)_{\alpha\in A}$ in E with $u_\alpha \downarrow 0$. Endow the set

$$B := \big\{(\alpha,\beta) :\, \alpha \in A,\, \beta \in [-u_\alpha, u_\alpha]\big\}$$

with the following partial order: $(\alpha',\beta') \le (\alpha'',\beta'')$ if and only if either $\alpha' < \alpha''$ or $\alpha' = \alpha''$ and $\beta' \le \beta''$. Obviously, B is a directed set. Now consider the net $\big(x_{(\alpha,\beta)}\big)_{(\alpha,\beta)\in B}$ defined by setting $x_{(\alpha,\beta)} = x_0 + \beta$ for all $(\alpha,\beta) \in B$. Set also $u'_{(\alpha,\beta)} := u_\alpha$ for all $(\alpha,\beta) \in B$. Since $u_\alpha \downarrow 0$, we have $u'_{(\alpha,\beta)} \downarrow 0$ as well. Since

$$\big(\forall(\alpha,\beta) \in B\big) \quad |x_{(\alpha,\beta)} - x_0| = |\beta| \le u_\alpha = u'_{(\alpha,\beta)},$$

one has $x_{(\alpha,\beta)} \xrightarrow{s.o} x_0$. By the order continuity of T at x_0, choose a net $\big(t_{(\alpha,\beta)}\big)_{(\alpha,\beta)\in B}$ in F and $(\alpha_0,\beta_0) \in B$ so that $t_{(\alpha,\beta)} \downarrow 0$ and

$$\big(\forall(\alpha,\beta) \ge (\alpha_0,\beta_0)\big) \quad \big|T(x_0+\beta) - T(x_0)\big| = \big|T(x_{(\alpha,\beta)}) - T(x_0)\big| \le t_{(\alpha,\beta)}.$$

Then

$$(\forall\alpha > \alpha_0)(\forall\beta \in [-u_\alpha, u_\alpha]) \quad \big|T(x_0+\beta) - T(x_0)\big| \le t_{(\alpha,\beta)} \le t(\alpha, -u_\alpha).$$

Hence,

$$(\forall\alpha > \alpha_0) \quad v_\alpha \le t_{(\alpha,-u_\alpha)}.$$

Since $t_{(\alpha,-u_\alpha)} \downarrow 0$, the latter inequality implies the first equality in (5.1).

(ii) ⇒ (i) is obvious. □

Given any nonempty set Ω, by $c_{00}(\Omega)$ we denote the Riesz space of all functions $f\colon \Omega \to \mathbb{R}$ with finite support, endowed with the natural order. Clearly, for every finite set Ω the uniform convergence in $c_{00}(\Omega)$ is equivalent to the order convergence. Thus, $c_{00}(\Omega) \in \mathcal{D}(F)$ for every Riesz space F.

Corollary 5.4. *Let Ω be an infinite set and $E \subseteq \mathbb{R}^\Omega$ be a Riesz space with $c_{00}(\Omega) \subseteq E$. Then $E \in \mathcal{D}(F)$ for every Riesz space F with the principal projection property.*

Proof. Let I be the directed set of all nonempty finite subsets $i \subseteq \Omega$ ordered by inclusion $i \le j \;\Leftrightarrow\; i \subseteq j$. For every $i \in I$ we set $E_i = \{f \in E : \operatorname{supp} f \subseteq i\}$ and use Theorem 5.6. □

Corollary 5.5. *Let μ be a σ-finite measure on a measure space (X,Σ,μ) and $p \in [0,\infty]$. Then $L_p(\mu) \in \mathcal{D}(F)$ for every Riesz space F with the principal projection property.*

Proof. It is enough to show that the Riesz space E fulfils the assumptions of Theorem 5.1. Let $(u_\alpha)_{\alpha\in A}$ be a net in E such that $u_\alpha \downarrow 0$. Using Egorov's theorem, we construct an increasing sequence $(X_n)_{n=1}^\infty$ of measurable sets $X_n \subseteq X$ such that $\mu(X \setminus \bigcup_{n=1}^\infty X_n) = 0$ and $u_\alpha|_{X_n} \rightrightarrows 0$ for every $n \in \mathbb{N}$. It remains to set $E_n = \{x \cdot \mathbf{1}_{X_n} : x \in E\}$ for every $n \in \mathbb{N}$. □

Theorem 5.6. *Let E be a Riesz space, F be a Riesz space with the principal projection property, I be a directed set and $(E_i)_{i\in I}$ be a family of projection bands E_i in E such that*

(1) *$E_i \in \mathcal{D}(F)$ for every $i \in I$;*
(2) *$E_i \subseteq E_j$ for every $i,j \in I$ with $i < j$;*

(3) $P_i(x) \xrightarrow{h} x$ *for every* $x \in E$*, where* P_i *is the order projection associated with* E_i*.*

Then $E \in \mathcal{D}(F)$*.*

Proof. If the directed set I has a maximal element i_0, then it follows from $(1)-(3)$ that $E = E_{i_0} \in \mathcal{D}(F)$.

Let I has no maximal element and $T : E \to F$ be an arbitrary Uryson operator $T : E \to F$ which is both horizontally-to-order continuous and uniformly-to-order continuous. For every $i \in I$ we set $T_i = T|_{E_i}$. Clearly, every T_i is an abstract Uryson operator which is both horizontally-to-order continuous and uniformly-to-order continuous. Thus, according to (1), every T_i is order continuous.

Fix an element $x_0 \in E$ and show that T is order continuous at x_0. Let $(u_\alpha)_{\alpha \in A}$ be a net in E such that $u_\alpha \downarrow 0$. For every $\alpha \in A$ we set

$$Y_\alpha = \{T(x) - T(x_0) : |x - x_0| \le u_\alpha\} \qquad \text{and} \qquad v_\alpha = \sup Y_\alpha.$$

According to Proposition 5.3, it enough to show that

$$v := \inf\{v_\alpha : \alpha \in A\} = 0.$$

For every $\alpha \in A$ and $i \in I$ we set

$$Y_{\alpha,i} = \{T(P_i(x)) - T(P_i(x_0)) : |x - x_0| \le u_\alpha\},$$

$$Z_{\alpha,i} = \bigcup_{j>i} \{T(P_j(x) - P_i(x)) - T(P_j(x_0) - P_i(x_0)) : |x - x_0| \le u_\alpha\},$$

$$Z_\alpha = \bigcup_{j \in I} Y_{\alpha,j},$$

$$v_{\alpha,i} = \sup Y_{\alpha,i}, \qquad w_{\alpha,i} = \sup Z_{\alpha,i} \qquad \text{and} \qquad w_\alpha = \sup Z_\alpha.$$

Claim 1. $w_\alpha = v_\alpha$ and consequently $w_\alpha \downarrow v$.

Since $Z_\alpha \subseteq Y_\alpha$, $w_\alpha \le v_\alpha$. It remains to show that $y \le w_\alpha$ for every $y \in Y_\alpha$. Let $y \in Y_\alpha$ be an arbitrary point and $x \in E$ such that $T(x) - T(x_0) = y$ and $|x - x_0| \le u_\alpha$. According to (3), $P_i(x) \xrightarrow{\mathrm{h}} x$ and $P_i(x_0) \xrightarrow{\mathrm{h}} x_0$. It follows from the horizontally-to-order continuity of T that

$$T(P_i(x)) \xrightarrow{\mathrm{o}} Tx \qquad \text{and} \qquad T(P_i(x_0)) \xrightarrow{\mathrm{o}} T(x_0).$$

Therefore,

$$z_i := T(P_i(x)) - T(P_i(x_0)) \xrightarrow{\mathrm{o}} T(x) - T(x_0) = y.$$

Since $z_i \in Y_{\alpha,i} \subseteq Z_\alpha$, $z_i \le w_\alpha$. Thus, $y \le w_\alpha$.

Claim 2. $Z_\alpha = Y_{\alpha,i} + Z_{\alpha,i}$ and $w_\alpha = v_{\alpha,i} + w_{\alpha,i}$.

Notice that according to (2) and [3, Theorem 1.46] for every $x \in E$ and $j > i$ we have that $P_i(x) \sqsubseteq P_j(x)$. Since T is orthogonally additive,

$$T(P_j(x)) - T(P_j(x_0)) = T(P_i(x)) - T(P_i(x_0)) + T(P_j(x) - P_i(x)) - T(P_j(x_0) - P_i(x_0)).$$

Therefore, $Z_\alpha \subseteq Y_{\alpha,i} + Z_{\alpha,i}$.

It remains to show that $Z_\alpha \supseteq Y_{\alpha,i} + Z_{\alpha,i}$. Let $y \in Y_{\alpha,i}$ and $z \in Z_{\alpha,i}$ be arbitrary elements. Then there exist $x_1, x_2 \in E$ and $j > i$ such that

$$|x_1 - x_0| \le u_\alpha, \qquad y = T(P_i(x_1)) - T(P_i(x_0)),$$

$$|x_2 - x_0| \le u_\alpha \qquad \text{and} \qquad z = T(P_j(x_2) - P_i(x_2)) - T(P_j(x_0) - P_i(x_0)).$$

Since $P_j(x'') - P_i(x'') \perp P_i(x')$ and T is orthogonally additive,

$$T(P_j(x'') - P_i(x'')) + T(P_i(x')) = T(P_j(x'') - P_i(x'') + P_i(x'))$$

for every $x', x'' \in E$. Therefore,

$$y + z = T(P_j(x_2) - P_i(x_2) + P_i(x_1)) - T(P_j(x_0).$$

We consider the element $x = (x_2 - P_i(x_2)) + P_i(x_1)$. Notice that $P_i(x) = P_i(x_1)$ and

$$P_j(x) = P_j(x_2) - P_j(P_i(x_2)) + P_j(P_i(x_1)) = P_j(x_2) - P_i(x_2) + P_i(x_1).$$

Therefore, in particular,

$$y + z = T(P_j(x)) - T(P_j(x_0).$$

Moreover,

$$Q_i(x) = Q_i(x_2),$$

where $Q_i(u) = u - P_i(u)$. Now we have

$$|x - x_0| = |P_i(x - x_0) + Q_i(x - x_0)| \le |P_i(x_1 - x_0)| + |Q_i(x_2 - x_0)| \le$$
$$\le P_i(u_\alpha) + Q_i(u_\alpha) = u_\alpha.$$

Thus, $y + z \in Y_{\alpha,j} \subseteq Z_\alpha$.

Claim 3. For every fixed $i \in I$ we have that $v_{\alpha,i} \downarrow 0$.

Since $u_\alpha \downarrow 0$, $P_i(u_\alpha) \downarrow 0$. It follows from the order continuity of T_i and Proposition 5.3 that $\tilde{v}_\alpha \downarrow 0$, where

$$\tilde{Y}_\alpha = \{T_i(x) - T_i(P_i(x_0)) : x \in E_i,\ |x - P_i(x_0)| \le P_i(u_\alpha)\} \qquad \text{and} \qquad \tilde{v}_\alpha = \sup \tilde{Y}_\alpha.$$

Notice that if $|x - x_0| \le u_\alpha$ then

$$|P_i(x) - P_i(x_0)| = |P_i(x - x_0)| \le P_i(u_\alpha).$$

Therefore, $Y_{\alpha,i} \subseteq \tilde{Y}_\alpha$, $v_{\alpha,i} \le \tilde{v}_\alpha$ and $v_{\alpha,i} \downarrow 0$.

Claim 4. $w_{\alpha,i} \ge v$.

Fix any $i \in I$. According to Claim 1-3, we have that $w_{\alpha,i} = w_\alpha - v_{\alpha,i}$, $w_\alpha \downarrow v$ and $v_{\alpha,i} \downarrow 0$. Thus, $w_{\alpha,i} \xrightarrow{\text{o}} v$. Moreover, $w_{\alpha,i} \downarrow$. Therefore,

$$w_{\alpha,i} \ge \inf\{w_{\alpha,j} : j \in I\} = v.$$

Claim 5. $v = 0$.

Fix any $\alpha \in A$. It is enough to show that the set $B = Z_\alpha$ satisfies the following condition from Lemma 5.2

$$\sup\{z \in B : y + z \in B\} \ge v$$

for every $y \in B$. Indeed, let $y \in Z_\alpha$ be an arbitrary element. We choose $i \in I$ such that $y \in Y_{\alpha,i}$. According to Claim 2 and Claim 4,

$$Z_{\alpha,i} \subseteq \{z \in B : y + z \in B\}$$

and

$$\sup\{z \in B : y + z \in B\} \ge w_{\alpha,i} \ge v.$$

□

Proof of Theorem 5.1. Let F be a Riesz space with the principal projection property and $T : E \to F$ be an arbitrary Uryson operator which is both horizontally-to-order continuous and uniformly-to-order continuous. Fix an element $x_0 \in E$ and show that T is order continuous at x_0. Let $(u_\alpha)_{\alpha\in A}$ be a net in E such that $u_\alpha \downarrow 0$. For every $\alpha \in A$ we set

$$Y_\alpha = \{T(x) - T(x_0) : |x - x_0| \le u_\alpha\} \qquad \text{and} \qquad v_\alpha = \sup Y_\alpha.$$

According to Proposition 5.3, it enough to show that

$$v := \inf\{v_\alpha : \alpha \in A\} = 0.$$

We choose a net $(E_i)_{i\in I}$ of projection bands E_i in E which satisfies conditions $(1)-(3)$. For every $i \in I$ we set $T_i = T|_{E_i}$. Clearly, every T_i is an abstract Uryson operator which is both horizontally-to-order continuous and uniformly-to-order continuous.

If the directed set I has a maximal element i_0, then it follows from $(1)-(3)$ that $E = E_{i_0}$, $T = T_{i_0}$ and $v = 0$.

Let I has no maximal element. For every $\alpha \in A$ and $i \in I$ we set

$$Y_{\alpha,i} = \{T(P_i(x)) - T(P_i(x_0)) : |x - x_0| \leq u_\alpha\},$$

$$Z_{\alpha,i} = \bigcup_{j>i}\{T(P_j(x) - P_i(x)) - T(P_j(x_0) - P_i(x_0)) : |x - x_0| \leq u_\alpha\},$$

$$Z_\alpha = \bigcup_{j\in I} Y_{\alpha,j},$$

$$v_{\alpha,i} = \sup Y_{\alpha,i}, \qquad w_{\alpha,i} = \sup Z_{\alpha,i} \qquad \text{and} \qquad w_\alpha = \sup Z_\alpha.$$

Claim 1. $w_\alpha = v_\alpha$ and consequently $w_\alpha \downarrow v$.

Since $Z_\alpha \subseteq Y_\alpha$, $w_\alpha \leq v_\alpha$. It remains to show that $y \leq w_\alpha$ for every $y \in Y_\alpha$. Let $y \in Y_\alpha$ be an arbitrary point and $x \in E$ such that $T(x) - T(x_0) = y$ and $|x - x_0| \leq u_\alpha$. According to (3), $P_i(x) \xrightarrow{\text{h}} x$ and $P_i(x_0) \xrightarrow{\text{h}} x_0$. It follows from the horizontally-to-order continuity of T that

$$T(P_i(x)) \xrightarrow{\text{o}} Tx \qquad \text{and} \qquad T(P_i(x_0)) \xrightarrow{\text{o}} T(x_0).$$

Therefore,

$$z_i := T(P_i(x)) - T(P_i(x_0)) \xrightarrow{\text{o}} T(x) - T(x_0) = y.$$

Since $z_i \in Y_{\alpha,i} \subseteq Z_\alpha$, $z_i \leq w_\alpha$. Thus, $y \leq w_\alpha$.

Claim 2. $Z_\alpha = Y_{\alpha,i} + Z_{\alpha,i}$ and $w_\alpha = v_{\alpha,i} + w_{\alpha,i}$.

Notice that according to (2) and [3, Theorem 1.46] for every $x \in E$ and $j > i$ we have that $P_i(x) \sqsubseteq P_j(x)$. Since T is orthogonally additive,

$$T(P_j(x)) - T(P_j(x_0)) = T(P_i(x)) - T(P_i(x_0)) + T(P_j(x) - P_i(x)) - T(P_j(x_0) - P_i(x_0)).$$

Therefore, $Z_\alpha \subseteq Y_{\alpha,i} + Z_{\alpha,i}$.

It remains to show that $Z_\alpha \supseteq Y_{\alpha,i} + Z_{\alpha,i}$. Let $y \in Y_{\alpha,i}$ and $z \in Z_{\alpha,i}$ be arbitrary elements. Then there exist $x_1, x_2 \in E$ and $j > i$ such that

$$|x_1 - x_0| \leq u_\alpha, \qquad y = T(P_i(x_1)) - T(P_i(x_0)),$$

$$|x_2 - x_0| \leq u_\alpha \qquad \text{and} \qquad z = T(P_j(x_2) - P_i(x_2)) - T(P_j(x_0) - P_i(x_0)).$$

Since $P_j(x'') - P_i(x'') \perp P_i(x')$ and T is orthogonally additive,

$$T(P_j(x'') - P_i(x'')) + T(P_i(x')) = T(P_j(x'') - P_i(x'') + P_i(x'))$$

for every $x', x'' \in E$. Therefore,

$$y + z = T(P_j(x_2) - P_i(x_2) + P_i(x_1)) - T(P_j(x_0).$$

We consider the element $x = (x_2 - P_i(x_2)) + P_i(x_1)$. Notice that $P_i(x) = P_i(x_1)$ and

$$P_j(x) = P_j(x_2) - P_j(P_i(x_2)) + P_j(P_i(x_1)) = P_j(x_2) - P_i(x_2) + P_i(x_1).$$

Therefore, in particular,

$$y + z = T(P_j(x)) - T(P_j(x_0)).$$

Moreover,

$$Q_i(x) = Q_i(x_2),$$

where $Q_i(u) = u - P_i(u)$. Now we have

$$|x - x_0| = |P_i(x - x_0) + Q_i(x - x_0)| \le |P_i(x_1 - x_0)| + |Q_i(x_2 - x_0)| \le$$
$$\le P_i(u_\alpha) + Q_i(u_\alpha) = u_\alpha.$$

Thus, $y + z \in Y_{\alpha,j} \subseteq Z_\alpha$.

Claim 3. For every fixed $i \in I$ we have that $v_{\alpha,i} \downarrow 0$.

Since $P_i(u_\alpha) \rightrightarrows 0$ and T_i is uniformly-to-order continuous at x_0, it follows from Proposition 5.3 that $\tilde{v}_\alpha \downarrow 0$, where

$$\tilde{Y}_\alpha = \{T_i(x) - T_i(P_i(x_0)) : x \in E_i,\ |x - P_i(x_0)| \le P_i(u_\alpha)\} \quad \text{and} \quad \tilde{v}_\alpha = \sup \tilde{Y}_\alpha.$$

Notice that if $|x - x_0| \le u_\alpha$ then

$$|P_i(x) - P_i(x_0)| = |P_i(x - x_0)| \le P_i(u_\alpha).$$

Therefore, $Y_{\alpha,i} \subseteq \tilde{Y}_\alpha$, $v_{\alpha,i} \le \tilde{v}_\alpha$ and $v_{\alpha,i} \downarrow 0$.

Claim 4. $w_{\alpha,i} \ge v$.

Fix any $i \in I$. According to Claim 1-3, we have that $w_{\alpha,i} = w_\alpha - v_{\alpha,i}$, $w_\alpha \downarrow v$ and $v_{\alpha,i} \downarrow 0$. Thus, $w_{\alpha,i} \xrightarrow{\text{o}} v$. Moreover, $w_{\alpha,i} \downarrow$. Therefore,

$$w_{\alpha,i} \ge \inf\{w_{\alpha,j} : j \in I\} = v.$$

Claim 5. $v = 0$.

Fix any $\alpha \in A$. It is enough to show that the set $B = Z_\alpha$ satisfies the following condition from Lemma 5.2

$$\sup\{z \in B : y + z \in B\} \ge v$$

for every $y \in B$. Indeed, let $y \in Z_\alpha$ be an arbitrary element. We choose $i \in I$ such that $y \in Y_{\alpha,i}$. According to Claim 2 and Claim 4,

$$Z_{\alpha,i} \subseteq \{z \in B : y + z \in B\}$$

and

$$\sup\{z \in B : y + z \in B\} \ge w_{\alpha,i} \ge v.$$

□

6. When does the order continuity of T imply that of $|T|$?

The following theorem gives a negative answer to Problem 2. Moreover, in the next section we provide an example which demonstrates that the assumption on E to have the principal projection property in Theorem 6.1 cannot be removed, or even replaced with the intersection property.

Theorem 6.1. *Let E be a Riesz space with the principal projection property, F a Dedekind complete Riesz space and $T \in \mathcal{P}(E, F)$. If T is order continuous then $|T|$ is order continuous.*

For the proof, we need the following lemma.

Lemma 6.2. *Let E be a Riesz space, $x, y \in E$ and $u \sqsubseteq x$. If u is a projection element then there exists $v \sqsubseteq y$ such that*

$$x - u \perp v, \quad y - v \perp u \quad \text{and} \quad u - v \sqsubseteq x - y.$$

Proof. Set

$$v := P_u y \overset{(2.1)}{=} \sup_n (y^+ \wedge n|u|) - \sup_n (y^- \wedge n|u|)$$

and prove that v possesses the desired properties. The relation $v \sqsubseteq y$ follows from (iii) of [3, Theorem 1.44].

Since $x - u \perp nu$ for all $n \in \mathbb{N}$, one has

$$|x-u| \wedge v^{\pm} = |x-u| \wedge \sup_n (y^{\pm} \wedge n|u|) = \sup_n (|x-u| \wedge y^{\pm} \wedge n|u|) = 0,$$

which implies $x - u \perp v$.

Observe that $|v| = P_u|y|$. Since $|y-v| \wedge |y| = |y-v|$, we obtain

$$\begin{aligned} 0 = |y-v| \wedge |v| &= |y-v| \wedge \sup_n (|y| \wedge n|u|) \\ &= \sup_n (|y-v| \wedge |y| \wedge n|u|) \geq |y-v| \wedge |u|, \end{aligned}$$

which yields $y - v \perp u$.

Since $x - u \perp u$ and $x - u \perp v$, one has $x - u \perp u - v$. Likewise, $y - v \perp v$ and $y - v \perp u$ imply $y - v \perp u - v$. Finally, the latter two conclusions give

$$(x-y) - (u-v) = (x-u) - (y-v) \perp u - v,$$

which yields $u - v \sqsubseteq x - y$. □

The following simple example shows that Lemma 6.2 is false without the assumption on u to be a projection element. Let $E = C[0,1]$, $x(t) = |t - 1/2|$ for all $t \in [0,1]$, $u(t) = 1/2 - t$ if $t \leq 1/2$ and $u(t) = 0$, if $t > 1/2$, and $y(t) = 1$ for all $t \in [0,1]$. Then y has two fragments 0 and y, none of which satisfies the requirements.

Proof of Theorem 6.1. Let $x \in E$ be an arbitrary point and $(x_\alpha)_{\alpha \in A}$ a net in E with $x_\alpha \xrightarrow{s.o} x$ in E. Let $(u_\alpha)_{\alpha \in A}$ be a net in E such that $u_\alpha \downarrow 0$ and $|x_\alpha - x| \leq u_\alpha$ for some $\alpha_0 \in A$ and all $\alpha \geq \alpha_0$. To prove the order continuity of $|T|$, by Theorem 1.3 it is enough to prove that

$$\big(T^+(x_\alpha) - T^+(x)\big)^+ \xrightarrow{o} 0. \tag{6.1}$$

In the case where A has a maximal element the proof is obvious. So assume that A has no maximal element. For every $\alpha \in A$ we set $z_\alpha = x_\alpha - x$ and

$$B_\alpha = \{(\alpha, z) : z \in \mathfrak{F}_{z_\alpha}\}$$

and endow the set $B = \bigsqcup_{\alpha \in A} B_\alpha$ with the lexicographic partial order $\leq$, that is

$$(\alpha, z) \leq (\alpha', z') \;\Leftrightarrow\; ((\alpha < \alpha') \text{ or } (\alpha = \alpha' \text{ and } z \sqsubseteq z')).$$

Clearly, $(B, \leq)$ is a directed set.

For any $\beta = (\alpha, z) \in B$ we set $x_\beta = x + z$ and show that the net $(x_\beta)_{\beta \in B}$ strongly order converges to x in E. For every $\beta = (\alpha, z) \in B$ we set $u_\beta = u_\alpha$. Since $u_\alpha \downarrow 0$, one has $u_\beta \downarrow 0$. Moreover,

$$|x_\beta - x| = |z| \leq |z_\alpha| = |x - x_\alpha| \leq u_\alpha = u_\beta$$

for every $\beta = (\alpha, z) \geq \beta_0 = (\alpha_0, 0)$.

By the order continuity of T at x, there exists a net $(v_\beta)_{\beta\in B}$ in F such that $v_\beta \downarrow 0$ and $|T(x_\beta) - T(x)| \le v_\beta$ for some $\beta_1 = (\alpha_1, x_1) \in B$ and all $\beta \ge \beta_1$.

For every $\alpha \in A$ we set $w_\alpha = v_{(\alpha,0)}$. Then $w_\alpha \ge w_{\alpha'}$ for all $\alpha, \alpha' \in A$ with $\alpha < \alpha'$. Since A has no maximal element, for every $\alpha \in A$ there exists $\alpha' \in A$ with $\alpha < \alpha'$, and for every $z \in \mathfrak{F}_{z_\alpha}$ we have that

$$v_{(\alpha,z)} \ge v_{(\alpha',0)} = w_{\alpha'}.$$

Therefore,

$$0 = \bigwedge_{\beta\in B} v_\beta = \bigwedge_{\alpha\in A} \bigwedge_{z\in\mathfrak{F}_{z_\alpha}} v_{(\alpha,z)} \ge \bigwedge_{\alpha\in A} w_\alpha.$$

Thus, $w_\alpha \downarrow 0$.

Let $\alpha_2 > \alpha_1$ be a fixed index and $\alpha \ge \alpha_2$ an arbitrary index. Let $s \sqsubseteq x_\alpha$ be an arbitrary fragment. By Lemma 6.2, there exists $t_s \sqsubseteq x$ such that

$$s - t_s \sqsubseteq x_\alpha - x = z_\alpha, \;\; s \perp x - t_s \;\text{ and }\; t_s \perp x - t_s$$

so $z := s - t_s \in \mathfrak{F}_{z_\alpha}$, $\beta = (\alpha, z) \ge \beta_1$ and

$$\begin{aligned} \big|T(s) - T(t_s)\big| &= \big|T(s) + T(x - t_s) - T(t_s) - T(x - t_s)\big| \\ &= \big|T(x + z) - T(x)\big| \\ &= \big|T(x_\beta) - T(x)\big| \le v_\beta \le v_{(\alpha,0)} = w_\alpha. \end{aligned}$$

Thus,

$$T(s) \le T(t_s) + w_\alpha$$

for every $s \sqsubseteq x_\alpha$. Hence,

$$\begin{aligned} T^+(x_\alpha) &= \sup\{T(s) : s \in \mathfrak{F}_{x_\alpha}\} \\ &\le \sup\{T(t_s) + w_\alpha : s \in \mathfrak{F}_{x_\alpha}\} \\ &\le \sup\{T(r) + w_\alpha : r \in \mathfrak{F}_x\} = T^+(x) + w_\alpha, \end{aligned}$$

which implies (6.1). □

The following proposition shows that, the order continuity of T at zero implies the order continuity of $|T|$ at zero without any assumption on E.

Proposition 6.3. *Let E and F be Riesz spaces with F Dedekind complete and $T \in \mathcal{OA}_r(E, F)$. If T is order continuous at 0 then $|T|$ is order continuous at 0.*

Proof. Let $(x_\alpha)_{\alpha\in A}$ be a in E with $x_\alpha \xrightarrow{s.o} 0$. We prove that $|T|(x_\alpha) \xrightarrow{o} 0$. In the case where A has a maximal element the proof is obvious. So assume that A has no maximal element. Let $(u_\alpha)_{\alpha\in A}$ be a net in E such that $|x_\alpha| \le u_\alpha$ for all $\alpha \ge \alpha_0$, where $\alpha_0 \in A$ and $u_\alpha \downarrow 0$. For every $\alpha \in A$ we set

$$B_\alpha = \{(\alpha, x) : x \in \mathfrak{F}_{x_\alpha}\}$$

and endow the set $B = \bigsqcup_{\alpha\in A} B_\alpha$ with the lexicographic partial order $\le$, that is

$$(\alpha, x) \le (\alpha', x') \;\; \Leftrightarrow \;\; ((\alpha < \alpha') \;\text{ or }\; (\alpha = \alpha' \;\text{ and }\; x \sqsubseteq x').$$

Clearly, $(B, \le)$ is a directed set.

For any $\beta = (\alpha, x) \in B$ we set $x_\beta = x$ and show that the net $(x_\beta)_{\beta\in B}$ strongly order converges to 0 in E. For every $\beta = (\alpha, x) \in B$ we set $u_\beta := u_\alpha$. Since $u_\alpha \downarrow 0$, one has $u_\beta \downarrow 0$. Moreover,

$$|x_\beta| \le |x_\alpha| \le u_\alpha = u_\beta$$

for every $\beta = (\alpha, x) \geq \beta_0 := (\alpha_0, 0)$.

It follows from the order continuity of T at 0 that there exists a net $(v_\beta)_{\beta \in B}$ in F such that $v_\beta \downarrow 0$ and $|T(x_\beta)| \leq v_\beta$ for some $\beta_1 = (\alpha_1, x_1) \in B$ and all $\beta \geq \beta_1$.

For every $\alpha \in A$ we set $w_\alpha = v_{(\alpha,0)}$. Observe that $w_\alpha \geq w_{\alpha'}$ for every $\alpha, \alpha' \in A$ with $\alpha < \alpha'$. Since A has no maximal element, for every $\alpha \in A$ there exists $\alpha' \in A$ with $\alpha < \alpha'$, and for every $x \in \mathfrak{F}_{x_\alpha}$ we have $v_{(\alpha,x)} \geq v_{(\alpha',0)} = w_{\alpha'}$. Therefore,

$$0 = \bigwedge_{\beta \in B} v_\beta = \bigwedge_{\alpha \in A} \bigwedge_{x \in \mathfrak{F}_{x_\alpha}} v_{(\alpha,x)} \geq \bigwedge_{\alpha \in A} w_\alpha.$$

Thus, $w_\alpha \downarrow 0$.

Let α_2 be a fixed index with $\alpha_2 > \alpha_1$ and $\alpha \geq \alpha_2$ be an arbitrary index. Then

$$\beta = (\alpha, x) \geq (\alpha, 0) \geq \beta_1$$

for every $x \in \mathfrak{F}_{x_\alpha}$. Therefore,

$$|T(x)| = |T(x_\beta)| \leq v_\beta \leq v_{(\alpha,0)} = w_\alpha$$

for every $x \in \mathfrak{F}_{x_\alpha}$, by Theorem 2.2, (3)

$$T^+(x_\alpha) = \sup\{Tx : x \in \mathfrak{F}_{x_\alpha}\} \leq w_\alpha.$$

Thus, $T^+(x_\alpha) \xrightarrow{o} 0$ in F. Since $|T| = T - 2T^+$ and $T(x_\alpha) \xrightarrow{o} 0$, we obtain $|T|(x_\alpha) \xrightarrow{o} 0$. ☐

Remark 6.4. As the above proof shows, the claim of Proposition 6.3 is valid if we replace the strong order convergence of nets in E with the weak order convergence.

7. An order continuous OAO with discontinuous modulus

In this section, we provide the first example of an order continuous abstract Uryson operator $T \in U(E, F)$ between Riesz spaces E, F with F Dedekind complete such that the modulus $|T|$ is not order continuous.

Given Riesz spaces E, F, G with $F \subseteq G$ and a mapping $f : E \to F$, by f^G we denote the same mapping $f^G : E \to G$. By $\boldsymbol{C}$ we denote the Dedekind completion of $C[0,1]$ in the sense that $C[0,1]$ is a majorizing order dense Riesz subspace of $\boldsymbol{C}$. By $\mathcal{O}(E, F)$ we denote the set of all OAOs $T : E \to F$, which is an ordered vector space with respect to the order $S \leq T$ if and only if $T - S \geq 0$. For $E = F$ we set $\mathcal{O}(E) := \mathcal{O}(E, E)$.

Theorem 7.1. *There exists a linear*[2] *order continuous operator* $S \in \mathcal{U}(C[0,1], \boldsymbol{C})$ *with order discontinuous* $|S|$.

At the first step, we construct an operator $T : C[0,1] \to C[0,1]$ with the same properties and then obtain the desired operator S as $S := T^{\boldsymbol{C}}$.

Theorem 7.2. *There exist linear order continuous order bounded operators* $T_1, T_2 : C[0,1] \to C[0,1]$ *such that*

1. *the moduli* $|T_1|$ *and* $|T_2|$ *exist in* $\mathcal{O}(C[0,1])$ *and are order continuous;*
2. *the operator* $T := T_1 + T_2$ *has the modulus* $|T|$ *in* $\mathcal{O}(C[0,1])$*;*
3. *the operator* $|T|$ *is not order continuous;*
4. *one has* $|T|^{\boldsymbol{C}} = |T^{\boldsymbol{C}}|$*, where by* $|T^{\boldsymbol{C}}|$ *we mean the modulus of* $T^{\boldsymbol{C}}$ *in the Dedekind complete Riesz space* $\mathcal{U}(C[0,1], \boldsymbol{C})$.

[2]Although S is linear, the modulus $|S|$ is considered as an OAO in $\mathcal{U}(C[0,1], \boldsymbol{C})$

To prove Theorem 7.2, we need some lemmas and preliminaries.

For every $x \in C[0,1]$ we set

$$a_x = \begin{cases} 0, & \text{if } x(t) \neq 0 \text{ on } [0, \frac{1}{2}] \\ \max\{t \in [0, \frac{1}{2}] : x(t) = 0\}, & \text{if } \{t \in [0, \frac{1}{2}] : x(t) = 0\} \neq \emptyset, \end{cases}$$

$$b_x = \begin{cases} 1, & \text{if } x(t) \neq 0 \text{ on } [\frac{1}{2}, 1] \\ \min\{t \in [\frac{1}{2}, 1] : x(t) = 0\}, & \text{if } \{t \in [\frac{1}{2}, 1] : x(t) = 0\} \neq \emptyset, \end{cases}$$

$$x_c = x \cdot \mathbf{1}_{[a_x, b_x]},$$

$$x_l = \begin{cases} 0, & \text{if } a_x = 0 \\ x \cdot \mathbf{1}_{[0, a_x]}, & \text{if } a_x \neq 0 \end{cases}$$

and

$$x_r = \begin{cases} 0, & \text{if } b_x = 1 \\ x \cdot \mathbf{1}_{[b_x, 1]}, & \text{if } b_x \neq 1. \end{cases}$$

The following two propositions are obvious.

Proposition 7.3. *For every $x \in C[0,1]$ one has*

1. *$x_l, x_c, x_r \in \mathfrak{F}_x$;*
2. *$x = x_l \sqcup x_c \sqcup x_r$.*

Proposition 7.4. *Let $x, y \in C[0,1]$ with $x \perp y$. Then*

1. *$x_l \perp y_l$ and $(x+y)_l = x_l + y_l$;*
2. *$x_r \perp y_r$ and $(x+y)_r = x_r + y_r$;*
3. *$(x+y)_c = x_c + y_c$, moreover, $x_c = 0$ or $y_c = 0$.*

Lemma 7.5. *Let $\varphi : [a,b] \to [c,d]$ be a strictly monotone continuous function and $T \colon C[c,d] \to C[a,b]$, $Tx(t) = x(\varphi(t))$. Then*

1. *T is a linear order continuous OAO;*
2. *if $x_1, x_2 \in C[c,d]$ and $x_1 \perp x_2$ then $Tx_1 \perp Tx_2$;*
3. *there exists the modulus $|T|$ of T in $\mathcal{O}\big(C[0,1]\big)$, which is order continuous.*

Proof. (1) is obvious. Verify (2). Let $x_1, x_2 \in C[c,d]$ and $x_1 \perp x_2$, $y_1 = Tx_1$. Set

$$A_1 = \{t \in [c,d] : x_1(t) \neq 0\} \qquad \text{and} \qquad A_2 = \{t \in [c,d] : x_2(t) \neq 0\}.$$

The condition $x_1 \perp x_2$ means that $A_1 \cap A_2 = \emptyset$. Since φ is strictly monotone,

$$\varphi^{-1}(A_1) \cap \varphi^{-1}(A_2) = \emptyset.$$

Therefore, $Tx_1 \perp Tx_2$.

To prove (3), consider the operator $S \colon C[c,d] \to C[a,b]$, $S(x) = |T(x)|$. It follows from (1) and (2) that S is an order continuous OAO. Moreover, $S = \sup\{T, -T\}$. Thus, $|T| = S$. □

Proof of Theorem 7.2. Consider the following operators $T_1, T_2 \colon C[0,1] \to C[0,1]$,

$$T_1 x(t) = x(\tfrac{t}{2}), \;\; t \in [0,1]$$

and

$$T_2 x(t) = -x(\tfrac{t+1}{2}), \;\; t \in [0,1].$$

By Lemma 7.5, T_1 and T_2 are linear order continuous order bounded operators which satisfy condition (1) of Theorem 7.2.

To prove (2), consider the operator $S \colon C[0,1] \to C[0,1]$,

$$S(x) = |T(x_l)| + |T(x_c)| + |T(x_r)|.$$

First we show that S is orthogonally additive. Notice that

$$T(x_l) = T_1(x_l) \qquad \text{and} \qquad T(x_r) = T_2(x_r)$$

for every $x \in C[0,1]$. Let $x, y \in C[0,1]$ and $x \perp y$. Condition (3) of Proposition 7.4 implies that $x_c = 0$ or $y_c = 0$. Suppose, for certainty, that $y_c = 0$. Now using Proposition 7.4 and Lemma 7.5 we obtain

$$\begin{aligned} S(x+y) &= |T(x+y)_l| + |T(x+y)_c| + |T(x+y)_r| \\ &= |T_1(x_l + y_l)| + |T(x_c)| + |T_2(x_r + y_r)| \\ &= |T_1(x_l)| + |T_1(y_l)| + |T(x_c)| + |T(y_c)| + |T_2(x_r)| + |T_2(y_r)| \\ &= S(x) + S(y). \end{aligned}$$

Now we prove that $S = \sup\{T, -T\}$. Since

$$T(x) = T(x_l) + T(x_c) + T(x_r) \leq |T(x_l)| + |T(x_c)| + |T(x_r)| = S(x)$$

and

$$-T(x) = -T(x_l) - T(x_c) - T(x_r) \leq |T(x_l)| + |T(x_c)| + |T(x_r)| = S(x)$$

for every $x \in C[0,1]$, we obtain

$$T \leq S \qquad \text{and} \qquad -T \leq S.$$

Let $R : C[0,1] \to C[0,1]$ be an orthogonally additive operator such that

$$T \leq R \qquad \text{and} \qquad -T \leq R.$$

Notice that $|Tx| \leq Rx$ for every $x \in C[0,1]$. Now we have that

$$\begin{aligned} S(x) &= |T(x_l)| + |T(x_c)| + |T(x_r)| \\ &\leq R(x_l) + R(x_c) + R(x_r) \\ &= R(x_l + x_c + x_r) = R(x) \end{aligned}$$

for every $x \in C[0,1]$. Thus, $S = |T|$. Moreover, by the above,

$$(\forall x \in C[0,1]) \;\; |T|(x) = |T(x_l)| + |T(x_c)| + |T(x_r)|. \tag{7.1}$$

(3) Let $x_n \in C[0,1]$ be the piece-wise linear function with nodes at the points $(0,0)$, $\left(\frac{1}{4^n}, 1\right)$, $\left(\frac{1}{2} - \frac{1}{4^n}, 1\right)$, $\left(\frac{1}{2}, 0\right)$, $\left(\frac{1}{2} + \frac{1}{4^n}, 1\right)$, $\left(1 - \frac{1}{4^n}, 1\right)$ and $(1,0)$ of $\mathbb{R}^2$, that is,

$$x_n(t) = \begin{cases} 1, & \text{if } t \in [\frac{1}{4^n}, \frac{1}{2} - \frac{1}{4^n}] \cup [\frac{1}{2} + \frac{1}{4^n}, 1 - \frac{1}{4^n}] \\ 4^n t, & \text{if } t \in [0, \frac{1}{4^n}] \\ \frac{4^n}{2} - 4^n t, & \text{if } t \in [\frac{1}{2} - \frac{1}{4^n}, \frac{1}{2}] \\ 4^n t - \frac{4^n}{2}, & \text{if } t \in [\frac{1}{2}, \frac{1}{2} + \frac{1}{4^n}] \\ 4^n - 4^n t, & \text{if } t \in [1 - \frac{1}{4^n}, 1]. \end{cases}$$

Notice that $(x_n)_{n=1}^\infty$ is strictly increasing and $\lim\limits_{n\to\infty} x_n(t) = 1$ for every $t \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. Therefore, $(x_n)_{n=1}^\infty$ order converges to $x_0 \equiv 1$ in $C[0,1]$.

On the other hand, $(x_0)_l = (x_0)_r = 0$,

$$S(x_0) = |T(x_0)| = 0,$$

$(x_n)_c = 0$ and

$$S(x_n)(t) = |T(x_n)_l| + |T(x_n)_r| = \begin{cases} 2, & \text{if } t \in [\frac{2}{4^n}, 1 - \frac{2}{4^n}] \\ 4^n t, & \text{if } t \in [0, \frac{2}{4^n}] \\ 4^n - 4^n t, & \text{if } t \in [1 - \frac{2}{4^n}, 1]. \end{cases}$$

for every $n \in \mathbb{N}$. Therefore, the sequence $(Sx_n)_{n=1}^{\infty}$ order converges to $y_0 \equiv 2$ in $C[0,1]$. Thus, S is not order continuous at x_0.

(4) We argue analogously as in (2) and prove that $|T|^{\boldsymbol{C}} = \sup\{T^{\boldsymbol{C}}, -T^{\boldsymbol{C}}\}$. For every $x \in C[0,1]$ one has $T^{\boldsymbol{C}}(x) = T(x) \le |T|(x) = |T|^{\boldsymbol{C}}(x)$ and hence $T^{\boldsymbol{C}} \le |T|^{\boldsymbol{C}}$. Analogously, $-T^{\boldsymbol{C}} \le |T|^{\boldsymbol{C}}$. Let $R : C[0,1] \to \boldsymbol{C}$ be an orthogonally additive operator such that

$$T^{\boldsymbol{C}} \le R \qquad \text{and} \qquad -T^{\boldsymbol{C}} \le R.$$

Since $|T^{\boldsymbol{C}}x| = |Tx| \le Rx$ for every $x \in C[0,1]$, we have that

$$|T|^{\boldsymbol{C}}(x) = |T(x_l)| + |T(x_c)| + |T(x_r)| \le R(x_l) + R(x_c) + R(x_r) =$$

$$R(x_l + x_c + x_r) = R(x)$$

for every $x \in C[0,1]$. Thus, $|T|^{\boldsymbol{C}} = |T^{\boldsymbol{C}}|$. □

Lemma 7.6. *Let E, F be Riesz spaces and G be the Dedekind completion of F in the sense that F is a majorizing order dense Riesz subspace of G.*

(1) *Any $T \in \mathcal{O}(E,F)$ is order bounded if and only if $T^G \in \mathcal{O}(E,G)$ is.*
(2) *Let $(y_\alpha)_{\alpha\in A}$ be a net in F and $y \in F$. Then $y_\alpha \xrightarrow{w-o} y$ in F if and only if $y_\alpha \xrightarrow{o} y$ in G.*

Proof. (1) Obviously, the order boundedness of T implies that of T^G, and the majorizing of F in G easily confirms the converse implication.

(2) is proved in [2, Theorem 1.7]. □

Proof of Theorem 7.1. Set $S := T^{\boldsymbol{C}}$, where T is defined in item (2) of Theorem 7.2. The order boundedness of S, as well as the order continuity of S and order discontinuity of $|S|$ follows from Theorem 7.2 and Lemma 7.6. □

References

1. N. Abasov, M. Pliev. On extensions of some nonlinear maps in vector lattices. J. Math. Anal. Appl. 455, no 1 (2017), 516-527. DOI: 10.1016/j.jmaa.2017.05.063
2. Yu. Abramovich, G. Sirotkin, *On order convergence of nets*, Positivity 9 (2005), no. 3, 287–292.
3. C. D. Aliprantis, O. Burkinshaw, *Positive Operators*, Springer, Dordrecht. (2006).
4. W. A. Feldman, *A factorization for orthogonally additive operators on Banach lattices*, J. Math. Anal. and Appl., 472 (2019), no. 1, 238–245.
5. O. Fotiy, I. Krasikova, M. Pliev, M. Popov. *Order continuity of orthogonally additive operators.* Results in Math. 77, no 5 (2022) published online. https://doi.org/10.1007/s00025-021-01543-x
6. A. Gumenchuk, O. Karlova, M. Popov. *Order Schauder bases in Banach lattices*, J. Funct. Anal. 269 (2) (2015), p. 536550. MR3348826, DOI: https://doi.org/10.1016/j.jfa.2015.04.008
7. A. Kamińska, I. Krasikova, M. Popov. *Projection lateral bands and lateral retracts*, Carpathian Math. Publ., 12 (2020), no. 2, 333–339. DOI: 10.15330/cmp.12.2.333-339
8. I. Krasikova, M. Pliev, M. Popov. *Measurable Riesz spaces.* Carpathian Math. Publ. 13, no 1 (2021), 81-88. DOI: 10.15330/cmp.13.1.81-88
9. V. Kadets, *A course in Functional Analysis and Measure Theory.* Translated from the Russian by Andrei Iacob. Universitext. Cham: Springer, 2018.
10. A. K. Kitover, A. W. Wickstead, *Operator norm limits of order continuous operators*, Positivity 9 (2005), no. 3, 341–355.
11. W. A. J. Luxemburg, A. C. Zaanen, *Riesz spaces.* Volume I. Elsevier, Amsterdam-London. (1970).
12. M. Martín, J. Merí, M. Popov, *On the numerical radius of operators in Lebesgue spaces*, J. Funct. Anal. 261 (1) (2011), 149168. MR2785896, DOI: 10.1016/j.jfa.2011.03.007

13. O. V. Maslyuchenko, V. V. Mykhaylyuk, M. M. Popov, *A lattice approach to narrow operators*, Positivity, 13 (2009), no. 3, 459–495.
14. J. M. Mazón, S. Segura de León, *Order bounded orthogonally additive operators*, Rev. Roumane Math. Pures Appl. 35, (1990), no. 4, 329–353.
15. J. M. Mazón, S. Segura de León, *Uryson operators*, Rev. Roumane Math. Pures Appl. 35, (1990), no. 5, 431–449.
16. V. Mykhaylyuk, M. Pliev, M. Popov, *The lateral order on Riesz spaces and orthogonally additive operators*, Positivity, 25 (2021), no. 2, 291–327. DOI: 10.1007/s11117-020-00761-x.
17. V. Mykhaylyuk, M. Pliev, M. Popov, *The lateral order on Riesz spaces and orthogonally additive operators. II*, preprint.
18. V. Orlov, M. Pliev, D. Rode, *Domination problem for AM-compact abstract Uryson operators*, Arch. Math., **107** (2016), 5, 543–552. DOI: 10.1007/s00013-016-0937-8
19. A. M. Plichko, M. M. Popov, *Symmetric function spaces on atomless probability spaces*, Dissertationes Math. (Rozprawy Mat.) 306 (1990), pp. 1–85.
20. M. Pliev, *Narrow operators on lattice-normed spaces*, Cent. Eur. J. Math. 9, No 6 (2011), pp. 1276–1287.
21. M. Pliev, *On C-compact orthogonally additive operators*, J. Math. Anal. Appl., 494 (2021), no. 1, 291–327. DOI: 10.1016/j.jmaa.2020.124594
22. M. Pliev, X. Fang, *Narrow orthogonally additive operators in lattice-normed spaces*, Sib. Math. J., **58** (2017), 1, 134-141.
23. M. A. Pliev, M. M. Popov, *Narrow orthogonally additive operators*, Positivity, 18 (2014), no. 4, 641–667.
24. M. A. Pliev, M. M. Popov, *On extension of abstract Uryson operators*, Siberian Math. J., 57 (2016), no. 3, 552-557.
25. M. Pliev, M. Popov, *Orthogonally additive operators on vector lattices*, Preprint (2022).
26. M. A. Pliev, K. Ramdane *Order unbounded orthogonally additive operators in vector lattices*, Mediter. J. Math., 15 (2018), no. 2, Paper No. 55. DOI 10.1007/s00009-018-1100-5
27. M. Popov, *Banach lattices of orthogonally additive operators*, to appear in J. Math. Anal. Appl.
28. M. Popov, B. Randrianantoanina, *Narrow Operators on Function Spaces and Vector Lattices*, De Gruyter Studies in Mathematics 45, Berlin-Boston, De Gruyter, 2013.

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