BIFURCATION OF CYCLES IN PARABOLIC SYSTEMS WITH WEAK DIFFUSION

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The existence of countably many cycles in hyperbolic systems of differential equations with transformed argument were considered in [6]. The existence and stability of an arbitrarily large finite number of cycles for the equation of spin combustion with delay were considered in [7]. We study the existence and stability of an arbitrarily large finite number of cycles for a parabolic system with delay and weak diffusion. Similar problems for partial differential equations were studied in numerous works (see, e.g., [1 − 7]).

1. Traveling waves for parabolic equations with weak diffusion. Consider a system

$$\frac{\partial u_1}{\partial t}=\varepsilon\gamma\frac{\partial^2 u_1}{\partial x^2}-\varepsilon\delta\frac{\partial^2 u_2}{\partial x^2}-\omega_0 u_2+\varepsilon(\alpha u_1-\beta u_2)+(d_0u_1-c_0u_2)(u_1^2+u_2^2),$$

$$\frac{\partial u_2}{\partial t}=\varepsilon\gamma\frac{\partial^2 u_2}{\partial x^2}+\varepsilon\delta\frac{\partial^2 u_1}{\partial x^2}+\omega_0 u_1+\varepsilon(\alpha u_2+\beta u_1)+(d_0u_2+c_0u_1)(u_1^2+u_2^2) \quad (1)$$

with periodic condition

$$u_1(t,x+2\pi)=u_1(t,x),\quad u_2(t,x+2\pi)=u_2(t,x), \quad (2)$$

where ε is a small positive parameter, $\omega_0>0$, $\alpha>0$, $\gamma>0$, $d_0<0$.

Passing to the complex variables $u=u_1+iu_2$ and $\bar u=u_1-iu_2$, we arrive at the equation

$$\frac{\partial u}{\partial t}=i\omega_0u+\varepsilon\left[(\gamma+i\delta)\frac{\partial^2 u}{\partial x^2}+(\alpha+i\beta)u\right]+(d_0+ic_0)u^2\overline{u}. \quad (3)$$

In the present paper, we investigate the existence and stability of the wave solutions of problem (1), (2). The solution of equation (3) is sought in the form of traveling wave $u=\theta(y)$, $y=\sigma t+x$, where the function $\theta(y)$ is periodic with period 2π. We arrive at the equation

$$\sigma\frac{d\theta}{dy}=i\omega_0\theta+\varepsilon\left[(\gamma+i\delta)\frac{d^2\theta}{dy^2}+(\alpha+i\beta)\theta\right]+(d_0+ic_0)\theta^2\overline{\theta}.$$

By the change of variables $\dfrac{d\theta}{dy}=\theta_1$, this equation is reduced to the following system:

$$\frac{d\theta}{dy}=\theta_1,\quad \sigma\theta_1=i\omega_0\theta+\varepsilon\left[(\gamma+i\delta)\frac{d\theta_1}{dy}+(\alpha+i\beta)\theta\right]+(d_0+ic_0)\theta^2\overline{\theta}. \quad (4)$$

The integral manifold of system (4) can be represented in the form

$$\theta_1=\frac{i\omega_0}{\sigma}\theta+\varepsilon\left[\frac{\alpha+i\beta}{\sigma}\theta-\frac{\omega_0^2}{\sigma^3}(\gamma+i\delta)\theta\right]+\frac{d_0+ic_0}{\sigma}\theta^2\bar\theta+\dots.$$

Here, we keep the terms of order $O(\varepsilon)$ in the linear terms and the terms of order $O(1)$ in the nonlinear terms. The equation on this manifold takes the form

$$\frac{d\theta}{dy}=\frac{i\omega_0}{\sigma}\theta+\varepsilon\left[\frac{\alpha+i\beta}{\sigma}\theta-\frac{\omega_0^2}{\sigma^3}(\gamma+i\delta)\theta\right]+\frac{d_0+ic_0}{\sigma}\theta^2\bar\theta+\dots. \quad (5)$$

Passing to the polar coordinates $\theta = r\exp(i\varphi)$ in Eq. (5), we get

$$\frac{dr}{dy} = \varepsilon\left(\frac{\alpha}{\sigma} - \frac{\gamma}{\sigma^3}\omega_0^2\right) r + \frac{d_0}{\sigma} r^3. \tag{6}$$

Let $d_0 < 0$ and let the inequality $\alpha > \dfrac{\gamma}{\sigma^2}\omega_0^2$ be true. Then Eq. (6) possesses the stationary solution

$$r = \sqrt{\varepsilon} R_0, \quad R_0 = \sqrt{\left(\alpha - \frac{\gamma}{\sigma^2}\omega_0^2\right)|d_0|^{-1}},$$

hence, the periodic solution of Eq. (5) takes the form $\theta = \sqrt{\varepsilon} R_0 \exp\left(\dfrac{i\omega_0}{\sigma} y\right) + O(\varepsilon)$. Since the function θ is periodic with period 2π, we get $\sigma = \dfrac{\omega_0}{n} + O(\varepsilon)$, $n = \pm 1, \pm 2, \ldots$. Thus, the periodic solution of Eq. (3) takes the form

$$u_n = u_n(t,x) = \sqrt{\varepsilon} r_n \exp(i(\chi_n(\varepsilon)t + nx)) + O(\varepsilon), \tag{7}$$

where $r_n = \sqrt{(\alpha - n^2\gamma)|d_0|^{-1}}$, $\chi_n(\varepsilon) = \omega_0 + \varepsilon\beta + \varepsilon c_0 r_n^2 - \varepsilon\delta n^2$, $n \in \mathbb{Z}$. Thus, the periodic solution of problem (1), (2) takes the form

$$u_1 = \sqrt{\varepsilon} r_n \cos(\chi_n(\varepsilon)t + nx), \quad u_2 = \sqrt{\varepsilon} r_n \sin(\chi_n(\varepsilon)t + nx), \quad n \in \mathbb{Z}. \tag{8}$$

The following statement is true:

Theorem 1. *Let $\omega_0 > 0$, $\alpha > 0$, $\gamma > 0$, $d_0 < 0$ and let the inequality $\alpha > \gamma n^2$ be true for some integer n. Then there exists $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$, problem (1), (2) has solutions (8) periodic in t.*

2. Stability of periodic solutions. The equation in variations in the vicinity of the solution (7) of equation (3) takes the form

$$\frac{\partial v}{\partial t} = i\omega_0 v + \varepsilon\left[(\gamma + i\delta)\frac{\partial^2 v}{\partial x^2} + (\alpha + i\beta)v\right] + \varepsilon(d_0 + ic_0)(2r_n^2 v + w_n^2\bar{v}), \tag{9}$$

where $w_n = r_n\exp(i(\chi_n(\varepsilon)t + nx))$, $\chi_n(\varepsilon) = \omega_0 + \varepsilon\beta + \varepsilon c_0 r_n^2 - \varepsilon\delta n^2$. By the change of variables $v = w\exp(i\chi_n(\varepsilon)t)$ in Eq. (9), we find

$$\frac{\partial w}{\partial t} = \varepsilon\left[(\gamma + i\delta)\frac{\partial^2 w}{\partial x^2} + \eta_n w + (d_0 + ic_0)r_n^2(w + \overline{w}\exp(2inx))\right], \tag{10}$$

where $\eta_n = \alpha + i\delta n^2 + d_0 r_n^2$.

We seek the solution of Eq. (10) in the form of Fourier series in the complex form

$$w(t,x) = \sum_{k=-\infty}^{\infty} y_k(t)\exp(ikx), \quad \overline{w}(t,x) = \sum_{k=-\infty}^{\infty} v_k(t)\exp(ikx). \tag{11}$$

Substituting (11) in (10) and equating the coefficients of $\exp(ikx)$, we obtain the equations for the coefficients of the Fourier series

$$\frac{dy_{k+n}}{dt} = \varepsilon[\eta_n y_{k+n} - (\gamma + i\delta)(k+n)^2 y_{k+n} + (d_0 + ic_0)r_n^2(y_{k+n} + v_{k-n})]. \tag{12}$$

Similarly, substituting (11) in the equation adjoint to (10), we get

$$\frac{dv_{k-n}}{dt} = \varepsilon[\overline{\eta}_n v_{k-n} - (\gamma - i\delta)(k-n)^2 v_{k-n} + (d_0 - ic_0)r_n^2(v_{k-n} + y_{k+n})]. \tag{13}$$

The stability of the wave solutions of problem (1), (2) is determined by the stability of system (12), (13) with a parameter $k \in \mathbb{Z}$. By the change of variables

$y_{k+n} = z_{k+n} \exp(-2i\varepsilon\delta kn)$, $v_{k-n} = w_{k-n} \exp(-2i\varepsilon\delta kn)$ in system (12), (13), we get a linear system with the matrix

$$\varepsilon A = \begin{pmatrix} \varepsilon a_{11} & \varepsilon a_{12} \\ \varepsilon a_{21} & \varepsilon a_{22} \end{pmatrix}.$$

The matrix A has an eigenvalue equal to zero for $k = 0$. Since the sum of diagonal elements of the matrix A is negative, $a = a_{11} + a_{22} < 0$, for the orbital exponential stability of the periodic solution $u_n(t, x)$, it is necessary and sufficient that the condition $a^2 c > f^2$, where $c = Re(det(A))$, $f = Im(det(A))$, $f = 4\gamma kn(c_0 r_n^2 - \delta k^2)$, be satisfied for $k \neq 0$, i.e.

$$(d_0 r_n^2 - \gamma k^2)^2(\gamma^2 k^2 + \delta^2 k^2 - 2\gamma d_0 r_n^2 - 4\gamma^2 n^2 - 2\delta c_0 r_n^2) > 4\gamma^2 n^2 (c_0 r_n^2 - \delta k^2)^2, \quad (14)$$

where $r_n^2 = (\gamma n^2 - \alpha)/d_0$.

Theorem 2. *The traveling waves $u_n(t, x)$ of problem (1), (2) are exponentially orbitally stable if and only if condition (14) is satisfied for all $k \in \mathbb{Z}\backslash\{0\}$.*

As an example, we consider a system (1), where $\delta = 0$, $c_0 = 0$. Hence, Theorem 1 implies that the periodic solution

$$u_n = \sqrt{\varepsilon(\alpha - \gamma n^2)|d_0|^{-1}} \begin{pmatrix} \cos(\omega_0 t + nx) \\ \sin(\omega_0 t + nx) \end{pmatrix}$$

exists for $d_0 < 0$ and $\gamma n^2 < \alpha$. By Theorem 2, the traveling waves $u_n(t, x)$ are exponentially orbitally stable if and only if $n^2 < \dfrac{1}{6\gamma}(\gamma + 2\alpha)$.

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