

## Approximation of solutions of boundary value problems for integro-differential equations of the neutral type using a spline function method

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**Abstract.** Boundary value problems for nonlinear integro-differential equations of the neutral type are investigated. A scheme for approximating the boundary value problem solution using cubic splines of defect two is proposed and substantiated. A model example illustrating the proposed approximation scheme is considered.

**2010 Mathematics Subject Classification:** Primary 34K10; Secondary 34K28

**Keywords:** boundary value problem, neutral type, cubic splines

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## Aproximarea soluțiilor problemelor cu valori la limită pentru ecuațiile integro-diferențiale de tip neutru folosind metoda funcției spline

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**Rezumat.** În lucrare sunt cercetate probleme cu valori la limită pentru ecuațiile integro-diferențiale neliniare de tip neutru. Este propusă și fundamentată o schemă de aproximare a soluției problemei cu valori la limită folosind spline cubice ale defectului doi. Se consideră un exemplu model care ilustrează schema de aproximare propusă.

**Cuvinte cheie:** problemă cu valoarea la limită, tip neutru, spline cubice

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In mathematical modeling of physical and technical processes, the evolution of which depends on prehistory, we arrive at differential equations with a delay. With the help of such equations it was possible to identify and describe new effects and phenomena in physics, biology, technology [1].

Boundary value problems for integro-differential equations with a deviating argument are mathematical models of various applied processes in biology, immunology, and medicine. In particular, Volterra integro-differential equations with a delay play an important role in modeling many real phenomena in ecology [2]. An important task in their study is to establish convenient conditions that guarantee the existence of solutions of such problems [2, 3]. Finding solutions to boundary value problems with a time delay in analytical form is possible only in the simplest cases, so the real task is to develop efficient methods for finding their approximate solutions [4]. The application of the spline function method to the approximation of boundary value problems for integro-differential

equations has been studied in [5]. Approximation schemes for differential-difference equations using systems of ordinary differential equations have been considered in [6, 7]. The aim of this work is to extend the approximation schemes using cubic splines of defect two for boundary value problems for integro-differential equations with many delays [8].

### 1. PROBLEM STATEMENT. SOLUTION EXISTENCE

Let us denote

$$\begin{aligned} [y(x)] &= \left( y(x - \tau_0(x)), \dots, y(x - \tau_n(x)) \right), \\ [y(x)]_1 &= \left( y'(x - \tau_0(x)), \dots, y'(x - \tau_n(x)) \right), \\ [y(x)]_2 &= \left( y''(x - \tau_0(x)), \dots, y''(x - \tau_n(x)) \right). \end{aligned} \quad (1)$$

Consider a boundary value problem

$$\begin{aligned} y''(x) &= f(x, [y(x)], [y(x)]_1, [y(x)]_2) + \\ &+ \int_a^b g(x, s, [y(s)], [y(s)]_1, [y(s)]_2) ds, \quad x \in [a; b], \\ y^{(p)}(x) &= \varphi^{(p)}(x), \quad p = 0, 1, 2, \quad x \in [a^*; a], \quad y(b) = \gamma, \end{aligned} \quad (2)$$

$$(3)$$

where  $\tau_0(x) = 0$ ,  $\tau_i(x)$ ,  $i = \overline{1, n}$  are continuous nonnegative functions defined on  $[a, b]$ ,  $\varphi(x)$  is a twice continuously differentiable function on  $[a^*; a]$ ,  $\gamma \in \mathbb{R}$ ,

$$a^* = \min_{0 < i \leq n} \left\{ \inf_{x \in [a; b]} (x - \tau_i(x)) \right\}.$$

Let us introduce the sets of points defined by the delays  $\tau_1(x), \dots, \tau_n(x)$ :

$$\begin{aligned} E_{i1} &= \{x_j \in [a, b] : x_j - \tau_i(x_j) = a, j = 1, 2, \dots\}, \\ E_{i2} &= \{x_j \in [a, b] : x_0 = a, x_{j+1} - \tau_i(x_{j+1}) = x_j, j = 0, 1, 2, \dots\}, \\ E_2 &= \bigcup_{i=1}^n (E_{i1} \cup E_{i2}). \end{aligned}$$

Assume that the delays  $\tau_i(x)$ ,  $i = \overline{1, n}$  are such functions that the sets  $E_{i1}, E_{i2}$ ,  $i = \overline{1, n}$  are finite. We will number the points of the set  $E_2$  in ascending order.

Let us introduce the notation:

$$P = \sup \left\{ \left| f(x, [y(x)], [y(x)]_1, [y(x)]_2) \right| + \left| \int_a^b g(x, s, [y(s)], [y(s)]_1, [y(s)]_2) ds \right| : \right.$$

$$\left. \begin{aligned} & \left| y(x - \tau_i(x)) \right| \leq P_1, \quad \left| y'(x - \tau_i(x)) \right| \leq P_2, \\ & \left| y''(x - \tau_i(x)) \right| \leq P_3, \quad i = \overline{0, n}, \quad x, s \in [a; b] \end{aligned} \right\},$$

$$J = [a^*; a], \quad I = [a, b],$$

$$I_1 = [a, x_1], \quad I_2 = [x_1, x_2], \quad \dots, \quad I_k = [x_{k-1}, x_k], \quad I_{k+1} = [x_k, b],$$

$$B_2(J \cup I) = \left\{ y(x) : y(x) \in \left( C(J \cup I) \cap \left( C^1(J) \cup C^1(I) \right) \cap \right.$$

$$\left. \cap \left( \bigcup_{j=1}^{k+1} C^2(I_j) \right) \right\}, \quad |y(x)| \leq P_1, \quad |y'(x)| \leq P_2, \quad |y''(x)| \leq P_3 \Big\},$$

where  $P_1, P_2, P_3$  are positive constants.

A function  $y = y(x)$  will be considered a solution of the boundary value problem (2)-(3) if it satisfies the equation (2) on  $[a; b]$  (with the possible exception of the points of the set  $E_2$ ) and the boundary conditions (3). We will find a solution of the problem (2)-(3) which belongs to the space  $B_2(J \cup I)$ .

The definition of the space  $B_2(J \cup I)$  implies that the solution of (2)-(3) is continuously differentiable for any  $x \in [a, b]$  where  $y'(a)$  is the right derivative, and at points of  $E_2$  there exist finite one-sided second derivatives of the solution which may not coincide.

Let us introduce a norm in the space  $B_2(J \cup I)$ :

$$\|y\|_{B_2} = \max \left\{ \frac{8}{(b-a)^2} \max_{x \in J \cup I} |y(x)|, \frac{2}{b-a} \max \left( \max_{x \in J} |y'(x)|, \max_{x \in I} |y'(x)| \right), \right.$$

$$\left. \max \left( \max_{x \in J} |y''(x)|, \max_{x \in I_1} |y''(x)|, \dots, \max_{x \in I_{k+1}} |y''(x)| \right) \right\}.$$

The space  $B_2(J \cup I)$  with this norm is a Banach space.

The boundary value problem (2)-(3) is equivalent to the integral equation [9]

$$y(x) = \int_{a^*}^b \left[ f(s, [y(s)], [y(s)]_1, [y(s)]_2) + \int_a^b g(s, \xi, [y(\xi)], [y(\xi)]_1, [y(\xi)]_2) d\xi \right] \times$$

$$\times \overline{G}(x, s) ds + l(x), \quad x \in J \cup I, \quad (4)$$

$$\bar{G}(x, s) = \begin{cases} G(x, s), & x, s \in I, \\ 0, & \text{otherwise,} \end{cases}$$

$$l(x) = \begin{cases} \varphi(x), & x \in J, \\ \frac{\gamma - \varphi(a)}{b-a}(x-a) + \varphi(a), & x \in I, \end{cases}$$

where  $G(x, s)$  is the Green's function of the boundary value problem

$$y''(x) = 0, \quad x \in I, \quad y(a) = y(b) = 0.$$

We define the operator  $T$  in the space  $B_2(J \cup I)$  as follows

$$(Ty)(x) = \int_{a^*}^b \left[ f(s, [y(s)], [y(s)]_1, [y(s)]_2) + \int_a^b g(s, \xi, [y(\xi)], [y(\xi)]_1, [y(\xi)]_2) d\xi \right] \bar{G}(x, s) ds + l(x), \quad x \in J \cup I.$$

Hence

$$(Ty)'(x) = \int_{a^*}^b \left[ f(s, [y(s)], [y(s)]_1, [y(s)]_2) + \int_a^b g(s, \xi, [y(\xi)], [y(\xi)]_1, [y(\xi)]_2) d\xi \right] \bar{G}'_x(x, s) ds + \frac{\gamma - \varphi(a)}{b-a},$$

$$x \in J \cup I. \tag{5}$$

$$(Ty)''(x) = f(x, [y(x)], [y(x)]_1, [y(x)]_2) + \int_a^b g(x, s, [y(s)], [y(s)]_1, [y(s)]_2) ds, \quad x \in J \cup I. \tag{6}$$

Let the function  $f(x, [y(x)], [y(x)]_1, [y(x)]_2)$  be continuous in  $G = [a; b] \times G_1^{n+1} \times G_2^{n+1} \times G_3^{n+1}$  and let  $g(x, s, [y(s)], [y(s)]_1, [y(s)]_2)$  be continuous in  $Q = [a; b] \times G$  where  $G_1 = \{u \in R : |u| < P_1\}$ ,  $G_2 = \{v \in R : |v| \leq P_2\}$ ,  $G_3 = \{w \in R : |w| \leq P_3\}$ ,  $P_1, P_2, P_3$  are positive constants that are included in the definition of the space  $B_2(J \cup I)$ .

The following theorem holds.

**Theorem 1.1.** *Let the conditions be met:*

- 1)  $\max \left\{ \max_{x \in J} |\varphi(x)|, \frac{(b-a)^2}{8} P + \max \left\{ |\varphi(a)|, |\gamma| \right\} \right\} \leq P_1,$
- 2)  $\max \left\{ \max_{x \in J} |\varphi'(x)|, \frac{b-a}{2} P + \left| \frac{\gamma - \varphi(a)}{b-a} \right| \right\} \leq P_2,$

$$3) \max \left\{ \max_{x \in J} |\varphi''(x)|, P \right\} \leq P_3,$$

4) the functions  $f(x, [y(x)], [y(x)]_1, [y(x)]_2)$  and  $g(x, s, [y(s)], [y(s)]_1, [y(s)]_2)$  satisfy the Lipschitz condition in  $G$  on the variables  $[y(x)], [y(x)]_1, [y(x)]_2$  with constants  $L_i^1$  and  $L_i^2$  ( $i = \overline{0, 3n+2}$ ), respectively,

$$5) \frac{(b-a)^2}{8} \sum_{i=0}^n \left( L_i^1 + (b-a)L_i^2 \right) + \frac{b-a}{2} \sum_{i=n+1}^{2n+1} \left( L_i^1 + (b-a)L_i^2 \right) + \sum_{i=2n+2}^{3n+2} \left( L_i^1 + (b-a)L_i^2 \right) < 1.$$

Then there exists a unique solution of the problem (2)-(3) in the space  $B_2(J \cup I)$ .

The proof is carried out similarly to Theorem 1 [5] using the contraction mapping principle.

## 2. COMPUTATIONAL SCHEME. ITERATIVE PROCESS CONVERGENCE

Choose an irregular grid  $\Delta = \{a = x_0 < x_1 < \dots < x_m = b\}$  on the interval  $[a; b]$  such that  $E_2 \subset \Delta$ . Let us denote by  $S(x, y)$  an interpolation cubic spline of defect two on  $\Delta$  for the function  $y(x)$ .  $S(x, y)$  belongs to the space  $B_2(J \cup I)$ .

We introduce the notation  $h_j = x_j - x_{j-1}$ ,  $j = 1, \dots, n$ ,  $M_j^+ = S''(x_j + 0, y)$ ,  $j = 0, \dots, n-1$ ,  $M_j^- = S''(x_j - 0, y)$   $j = 1, \dots, n$ . It is easy to obtain an image for the spline  $S(x, y)$ :

$$\begin{aligned} S(x, y) = & M_{j-1}^+ \frac{(x_j - x)^3}{6h_j} + M_j^- \frac{(x - x_{j-1})^3}{6h_j} + \\ & + \left( y_{j-1} - \frac{M_{j-1}^+ h_j^2}{6} \right) \frac{x_j - x}{h_j} + \left( y_j - \frac{M_j^- h_j^2}{6} \right) \frac{x - x_{j-1}}{h_j}, \\ & x \in [x_{j-1}; x_j], \quad j = 1, 2, \dots, m. \end{aligned} \quad (7)$$

Taking into account the form of the spline (7) and the continuity of its first derivatives in the internal nodes of the grid  $\Delta$  we obtain a system of linear algebraic equations satisfied by the values  $M_{j-1}^+$  and  $M_j^-$  ( $j = 1, 2, \dots, m$ ):

$$\begin{cases} h_{j+1}y_{j-1} - (h_j + h_{j+1})y_j + h_j y_{j+1} = \frac{h_j h_{j+1}}{6} \times \\ \times \left( h_j M_{j-1}^+ + 2h_j M_j^- + 2h_{j+1} M_j^+ + h_{j+1} M_{j+1}^- \right), \\ j = \overline{1, m-1}. \end{cases} \quad (8)$$

We will find a solution of the boundary value problem (2)-(3) in the form of a sequence of cubic splines with defect 2 according to the following scheme:

A) Choose an initial cubic spline  $S(x, y^{(0)}) = \frac{y - \varphi(a)}{b-a} (x - a) + \varphi(a)$  which satisfies the boundary conditions (3) at  $x = a$  and  $x = b$ .

B) Using the original equation (2) and the spline  $S(x, y^{(k)})$ , for  $k = 0, 1, \dots$  find:

$$M_j^{+(k+1)} = f(x_j, [S(x_j + 0, y^{(k)})], [S(x_j + 0, y^{(k)})]_1, [S(x_j + 0, y^{(k)})]_2) + \int_a^b g(x_j, s, [S(s, y^{(k)})], [S(s + 0, y^{(k)})]_1, [S(s + 0, y^{(k)})]_2) ds, \\ j = \overline{0, m-1}, \quad (9)$$

$$M_j^{-(k+1)} = f(x_j, [S(x_j - 0, y^{(k)})], [S(x_j - 0, y^{(k)})]_1, [S(x_j - 0, y^{(k)})]_2) + \int_a^b g(x_j, s, [S(s, y^{(k)})], [S(s - 0, y^{(k)})]_1, [S(s - 0, y^{(k)})]_2) ds, \\ j = \overline{1, m}. \quad (10)$$

In the correlations (9), (10) substitute  $S^{(p)}(x, y^{(k)}) = \varphi^{(p)}(x)$ ,  $p = 0, 1, 2$  for  $x < a$ .

C) Calculate  $y_j^{(k+1)}$ ,  $j = \overline{0, m}$  by solving the system of equations (8).

D) Obtain the cubic spline  $S(x, y^{(k+1)})$  in the form (7) using the previously calculated values  $y_j^{(k+1)}$ ,  $j = \overline{0, m}$ ,  $M_j^{+(k+1)}$ ,  $j = \overline{0, m-1}$ ,  $M_j^{-(k+1)}$ ,  $j = \overline{1, m}$ . This spline is the next iteration approximation.

Let us introduce the notation:

$$\lambda_1 = \sum_{i=0}^n (L_i^1 + (b-a)L_i^2), \quad (11) \\ \lambda_2 = \sum_{i=n+1}^{2n+1} (L_i^1 + (b-a)L_i^2), \quad \lambda_3 = \sum_{i=2n+2}^{3n+2} (L_i^1 + (b-a)L_i^2), \\ u = \frac{K^5}{8}(b-a)^2 + \frac{H^2}{8}, \quad v = \frac{K^5}{2}(b-a) + \frac{2H}{3}, \\ \mu = 5 \left( 1 + \frac{1}{2}\lambda_1 H^2 + \lambda_2 H + \lambda_3 \right).$$

**Theorem 2.1.** Assume that there exists a solution of the boundary value problem (2)-(3) and it belongs to the space  $B_2(J \cup I)$ . When the following inequality is true

$$\theta = u\lambda_1 + v\lambda_2 + \lambda_3 < 1, \quad (12)$$

then there exists  $H^*$  such that for all  $0 < H < H^*$  the sequence of splines  $\{S(x, y^{(k)})\}$ ,  $k = 0, 1, \dots$  converges uniformly on  $[a; b]$  and the following correlations hold

$$\left\| \lim_{k \rightarrow \infty} S^{(p)}(x, y^{(k)}) - y^{(p)}(x) \right\| \leq R_p \omega(y''(x), H), \quad p = 0, 1, 2, \quad (13)$$

$$R_0 = \sup_{H \leq H^*} \left( \frac{u\mu}{1-\theta} + \frac{5H^2}{2} \right), \quad R_1 = \sup_{H \leq H^*} \left( \frac{v\mu}{1-\theta} + 5H \right),$$

$$R_2 = \sup_{H \leq H^*} \left( \frac{\mu}{1-\theta} + 5 \right),$$

$$\omega(y''(x), H) = \max_{1 \leq r \leq l+1} \omega_r(y''(x), H),$$

where  $\omega_r(f, H)$  is the continuity modulus of the function  $f$  on the interval  $\delta_r$ .

### 3. EXAMPLE

Consider the boundary value problem for the neutral type equation:

$$y''(x) = \frac{1}{4}y''(x-1) + 1, \quad x \in [0; 2],$$

$$y(x) = x, \quad y'(x) = 1, \quad y''(x) = 0, \quad x \in [-1; 0], \quad y(2) = \frac{5}{2}.$$

The precise solution  $y(x)$  was found using the method of steps. The approximate solution  $y_S^{20}(x)$  and  $y_S^{40}(x)$ , according to the iterative scheme proposed in the work, was obtained on the 2nd iteration with a 20 and 40 segment grid respectively.  $\Delta_S^{20}$  and  $\Delta_S^{40}$  are the absolute errors of the approximate solutions.

**Table 1.** Precise and approximate solutions

$x$	$y(x)$	$y_S^{20}(x)$	$\Delta_S^{20}$	$y_S^{40}(x)$	$\Delta_S^{40}$
0.5	0.21875	0.21552	0.00323	0.21716	0.00159
1	0.6875	0.68146	0.00604	0.68443	0.00307
1.5	1.4375	1.43448	0.00302	1.43596	0.00154

When comparing the exact and approximate solutions, one can notice that the absolute error at 20 segments does not exceed 0.006, and the relative error – 0.8%. But at 40 segments the absolute error does not exceed 0.003, and the relative error – 0.4%.

### 4. CONCLUSION

In this paper we investigate boundary value problems for nonlinear integro-differential equations of neutral type. Sufficient conditions for the existence of solutions of such

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problems are established. The iterative schemes for finding approximate solutions of these problems using cubic splines of defect two are constructed and substantiated, the convergence of the iterative process is investigated. The use of the apparatus of spline functions allows us to construct algorithms that are simple to implement and at the same time suitable for solving a wide class of boundary value problems.

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