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ONE-SIDED BOUNDARY-VALUE PROBLEM WITH IMPULSIVE CONDITIONS FOR PARABOLIC EQUATIONS WITH DEGENERATION

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For a second-order parabolic equation, we consider a one-sided boundary-value problem with impulsive conditions with respect to the time variable. The coefficients of equation and the boundary conditions have power singularities of any order in time and space variables on a certain set of points. We also establish conditions for the existence and uniqueness of solution to the posed problem in Hölder spaces with power weights.

Keywords: parabolic equation with degeneration, power singularity, impulsive condition, existence and uniqueness of solutions.

Investigations of the problems of mechanics, theory of elasticity and control lead to the solution of one-sided boundary-value problems for differential equations. In particular, the monograph [2] is devoted to the analysis of these problems. The mathematical simulation of numerous physical and chemical phenomena leads to problems with degenerations and singularities for partial differential equations. In particular, the coefficients of the Schrödinger equation that describes the state of a quantum-mechanical system specify the potential energy and have power singularities at lower derivatives [3]. Problems of existence and the qualitative properties of solutions of boundary-value problems for equations with degenerations were studied in [4, 6, 9].

The investigation of systems with discontinuous trajectories is connected with development of a technique in which impulsive control systems play an important role. In the analysis of the problems of the theory of automatic control, theory of nuclear reactors, and dynamical systems, it is necessary to solve nonlocal boundary-value problems for equations with impulsive action. Problems for systems of impulsive ordinary differential equations were comprehensively studied in the works by Samoilenko and Perestyuk [7, 8, 10] and other researchers.

Problems of existence of nonperiodic or periodic solutions of impulsive hyperbolic partial differential equations were studied in [1, 7]. The second part of the monograph [5] was devoted to the construction of the theory of well-posedness of the Cauchy problem for parabolic systems with impulsive action in the Dini spaces.

In the present work, we consider a one-sided boundary-value problem with impulsive conditions for a linear parabolic equation with power singularities of any order in the coefficients of the equation and in the boundary condition with respect to any variables on a certain set of points. We prove the existence of a unique solution of the posed problem and establish estimates for its derivatives in Hölder spaces with power weights.

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1. Statement of the Problem and Main Result

Let η, t_0, t_1, …, t_N, t_{N+1} be fixed positive numbers, $0 \le t_0 < t_1 < \ldots < t_{N+1}$, $t_0 < \eta < t_{N+1}$, $\eta \ne t_\lambda$, $\lambda \in \{1,2,\ldots,N\}$, let D be a bounded domain in $\mathbb{R}^n$ with boundary ∂D, $\dim D = n$, and let Ω be a bounded domain, $\bar{\Omega} \subset D$, $\dim \Omega \le n-1$.

Denote

$$Q_{(0)} = \{(t,x) : t \in [t_0, t_{N+1}), x \in \Omega\} \cup \{(t,x) : t = \eta, x \in D\},$$

$$Q^{(k)} = [t_k, t_{k+1}) \times D, \qquad \Gamma^{(k)} = [t_k, t_{k+1}) \times \partial D, \qquad k \in \{0,1,\ldots,N\}.$$

In the domain $Q = [t_0, t_{N+1}) \times D$, we consider the problem of finding a function $u(t,x)$ satisfying, for $t \ne t_k$, $(t,x) \notin Q_{(0)}$, the following equation:

$$(Lu)(t,x) \equiv \left[\partial_t - \sum_{i,j=1}^{n} A_{ij}(t,x)\partial_{x_i}\partial_{x_j} + \sum_{i=1}^{n} A_i(t,x)\partial_{x_i} + A_0(t,x)\right] u(t,x) = f(t,x), \tag{1}$$

conditions with respect to the variable t:

$$u(t_0 + 0, x) = \varphi_0(x), \tag{2}$$

$$u(t_\lambda + 0, x) - u(t_\lambda - 0, x) = \psi_\lambda(x) u(t_\lambda - 0, x) + \varphi_\lambda(x), \tag{3}$$

and the boundary conditions

$$\lim_{x \to z \in \partial D} (Bu - g)(t,x) \equiv \lim_{x \to z \in \partial D} \left[\sum_{k=1}^{n} b_k(t,x)\partial_{x_k} u + b_0(t,x) u - g(t,x)\right] \ge 0, \tag{4}$$

$$\lim_{x \to z \in \partial D} u \ge 0, \qquad \lim_{x \to z \in \partial D} [u(Bu - g)](t,x) = 0 \tag{5}$$

on the lateral surface $\Gamma = [t_0, t_{N+1}) \times \partial D$.

The power singularities of coefficients of the differential expressions L and B at the point $P(t,x) \in Q \backslash Q_0$ are characterized by the functions $s_1(\beta_i^{(1)}, t)$ and $s_2(\beta_i^{(2)}, x)$:

$$s_1(\beta_i^{(1)}, t) = \begin{cases} |t-\eta|^{\beta_i^{(1)}}, & |t-\eta| \le 1, \\ 1, & |t-\eta| \ge 1, \end{cases} \qquad s_2(\beta_i^{(2)}, x) = \begin{cases} \rho^{\beta_i^{(2)}}(x), & \rho(x) \le 1, \\ 1, & \rho(x) \ge 1, \end{cases}$$

$$\rho(x) = \inf_{z \in \Omega} |x - z|, \qquad \beta_i^{(\nu)} \in (-\infty, \infty), \qquad \nu \in \{1,2\},$$

$$\beta^{(\nu)} = (\beta_1^{(\nu)}, \ldots, \beta_n^{(\nu)}), \qquad \beta = (\beta^{(1)}, \beta^{(2)}).$$

We now define spaces used to study problem (1)–(3). We introduce the notation: ℓ, $q^{(1)}$, $q^{(2)}$, $\gamma^{(1)}$, $\gamma^{(2)}$, $\delta^{(1)}$, $\delta^{(2)}$, $\mu_j^{(1)}$, $\mu_j^{(2)}$, $j \in \{0,1,\ldots,n\}$, are real numbers, $\ell \geq 0$, $[\ell]$ is the integral part of a number ℓ, $\{\ell\} = \ell - [\ell]$, $q^{(\nu)} \geq 0$, $\gamma^{(\nu)} \geq 0$, $\delta^{(\nu)} \geq 0$, $\mu_j^{(\nu)} \geq 0$, $\nu \in \{1,2\}$; $P(t,x)$, $P_1(t^{(1)},x^{(1)})$, $P_2(t^{(2)},x^{(1)})$, $R_i(t^{(1)},x^{(2)})$, $i \in \{1,2,\ldots,n\}$, are arbitrary points from $Q^{(k)}$, $x^{(1)} = (x_1^{(1)},\ldots,x_i^{(1)},\ldots,x_n^{(1)})$, and $x^{(2)} = (x_1^{(1)},\ldots,x_{i-1}^{(1)},x_i^{(2)},x_{i+1}^{(1)},\ldots,x_n^{(1)})$.

By $H^{\ell}(\gamma;\beta;q;Q)$ we denote the set of functions u with continuous derivatives in $Q^{(k)} \backslash Q_{(0)}$ for $t \neq t_k$ of the form $\partial_t^s \partial_x^r$, $2s + |r| \leq [\ell]$, and a finite norm

$$\|u;\gamma;\beta;0;Q\|_0 = \sup_k \{\sup_{\bar{Q}^{(k)}} |u|\} \equiv \|u;Q\|_0,$$

$$\|u;\gamma;\beta;q;Q\|_\ell = \sup_k \left\{ \sum_{2s+|r|\leq[\ell]} \left\|u;\gamma;\beta;q;Q^{(k)}\right\|_{2s+|r|} + \left\langle u;\gamma;\beta;q;Q^{(k)} \right\rangle_\ell \right\},$$

where, e.g.,

$$\left\|u;\gamma;\beta;q;Q^{(k)}\right\|_{2s+|r|} \equiv \sup_{P\in\bar{Q}^{(k)}} \Bigg[s_1(q^{(1)} + 2s\gamma^{(1)},t) s_2(q^{(2)} + 2s\gamma^{(2)},x)$$

$$\times \left|\partial_t^s \partial_x^r u(P)\right| \prod_{i=1}^n s_1(r_i(\gamma^{(1)} - \beta_i^{(1)}),t) s_2(r_i(\gamma^{(2)} - \beta_i^{(2)}),x) \Bigg],$$

$$\left\langle u;\gamma,\beta,q;Q^{(k)} \right\rangle_\ell \equiv \sum_{2s+|r|=[\ell]} \Bigg\{ \sum_{\nu=1}^n \Bigg[\sup_{(P_2,R_\nu)\subset\bar{Q}_k} \Bigg[s_1(q^{(1)} + 2s\gamma^{(1)},t^{(2)})$$

$$\times s_2(q^{(2)} + 2s\gamma^{(2)},\tilde{x}) \prod_{i=1}^n s_1(r_i(\gamma^{(1)} - \beta_i^{(1)}),t^{(2)}) s_2(r_i(\gamma^{(2)} - \beta_i^{(2)}),\tilde{x})$$

$$\times \left|\partial_t^s \partial_x^r u(P_2) - \partial_t^s \partial_x^r u(R_\nu)\right| \left|x_\nu^{(1)} - x_\nu^{(2)}\right|^{-\{\ell\}}$$

$$\times s_1(\{\ell\}(\gamma^{(1)} - \beta_\nu^{(1)}),t^{(2)}) s_2(\{\ell\}(\gamma^{(2)} - \beta_\nu^{(2)}),\tilde{x}) \Bigg]$$

$$+ \sup_{(P_1,P_2)\subset\bar{Q}_k} \Bigg[s_1(q^{(1)} + \ell\gamma^{(1)},\tilde{t}) s_2(q^{(2)} + (2s + \{\ell\})\gamma^{(2)},x^{(1)})$$

$$\times \prod_{i=1}^n s_1(-r_i\beta_i^{(1)},\tilde{t}) s_2(r_i(\gamma^{(2)} - \beta_i^{(2)}),x^{(1)}) \left|t^{(1)} - t^{(2)}\right|^{-\{\ell/2\}}$$

$$\times \left| \partial_t^s \partial_x^r u(P_1) - \partial_t^s \partial_x^r u(P_2) \right| \Bigg]\Bigg] \Bigg\}, \quad |r| = r_1 + \ldots + r_n,$$

$$s_1(q,\tilde{t}) = \min\{s_1(q,t^{(1)}), s_1(q,t^{(2)})\},$$

$$s_2(q,\tilde{x}) = \min\{s_2(q,x^{(1)}), s_2(q,x^{(2)})\}.$$

By $\Gamma^{(1)}$ we denote the set of points of the boundary Γ for which the following condition is satisfied:

$$\lim_{x \to z \in \partial D} u(t,x) = 0.$$

Thus, it follows from the boundary condition (5) that the condition

$$\lim_{x \to z \in \partial D} (Bu - g)(t,x) = 0$$

is satisfied at the points $\Gamma^{(2)} = \Gamma \backslash \Gamma^{(1)}$.

Assume that problem (1)–(3) satisfies the conditions:

$1°$. For any vector $\xi = (\xi_1, \ldots, \xi_n)$ and $\forall (t,x) \in Q \backslash Q_{(0)}$, the inequality

$$\pi_1 |\xi|^2 \le \sum_{i,j=1}^{n} A_{ij}(t,x) s_1(\beta_i^{(1)},t) s_1(\beta_j^{(1)},t) s_2(\beta_i^{(2)},x) s_2(\beta_j^{(2)},x) \xi_i \xi_j \le \pi_2 |\xi|^2 \qquad (6)$$

is true. Here, π_1 and π_2 are fixed positive constants,

$$s_1(\mu_i^{(1)},t) s_2(\mu_i^{(2)},x) A_i \in H^{\alpha}(\gamma;\beta;0;Q),$$

$$s_1(\mu_0^{(1)},t) s_2(\mu_0^{(2)},x) A_0 \in H^{\alpha}(\gamma;\beta;0;Q), \qquad A_0 \ge 0,$$

$$s_1(\delta^{(1)},t) s_2(\delta^{(2)},x) b_0 \in H^{1+\alpha}(\gamma;\beta;0;Q),$$

$$s_1(\beta_i^{(1)},t) s_1(\beta_j^{(1)},t) s_2(\beta_i^{(2)},x) s_2(\beta_j^{(2)},x) A_{ij} \in H^{\alpha}(\gamma;\beta;0;Q),$$

$$s_1(\beta_i^{(1)},t) s_2(\beta_i^{(2)},x) b_i \in H^{1+\alpha}(\gamma;\beta;0;Q)$$

and the vectors $\mathbf{b}^{(s)} = \{b_1^{(s)}, \ldots, b_n^{(s)}\}$, $b_i^{(s)} = s_1(\beta_i^{(1)},t) s_2(\beta_i^{(2)},x) b_i$, and $\mathbf{e} = \{e_1, \ldots, e_n\}$,

$$e_i = b_i \left(\sum_{k=1}^{n} b_k^2 \right)^{-1/2}$$

form the angles with the direction of the outer normal $\mathbf{n}$ to ∂D at the point $P(t,x) \in \Gamma^{(2)}$ that are smaller than $\pi/2$, $b_0(t,x)|_{\Gamma^{(2)}} > 0$, and $\partial\Omega \subset C^{2+\alpha}$.

$2°$. $f \in H^{\alpha}(\gamma;\beta;\mu_0;Q)$, $\varphi_k \in H^{2+\alpha}(\tilde{\gamma};\tilde{\beta},0;Q \cap (t=t_k))$, $\tilde{\gamma} = (0,\gamma^{(2)})$,

$$\tilde{\beta} = (0,\beta^{(2)}), \qquad g \in H^{1+\alpha}(\gamma;\beta;\delta;Q^{(k)}), \qquad \sum_{k=1}^{n} b_k(t,x)\frac{\partial \psi_\lambda}{\partial x_k}\bigg|_{\Gamma^{(2)}} = 0,$$

$$[g(t_\lambda + 0,x) - g(t_\lambda - 0,x)(1+\psi_\lambda(x)) - B\varphi_\lambda(x)]\big|_{\Gamma^{(2)}} = 0,$$

$$\psi_\lambda \in C^{2+\alpha}(Q \cap (t=t_\lambda)), \qquad \varphi_\lambda(x)\big|_{\Gamma^{(1)}} = 0,$$

$$\gamma^{(\nu)} = \max\left\{\max_i (1+\beta_i^{(\nu)}), \max_i (\gamma^{(\nu)} - \beta_i^{(\nu)}), \frac{\mu_0^{(\nu)}}{2}, \delta^{(\nu)}\right\}, \qquad \nu \in \{1,2\}.$$

The following theorem is true:

Theorem 1. *Suppose that problem (1)–(5) satisfies conditions 1° and 2°. Then there exists a unique solution of problem (1)–(5) from the space* $H^{2+\alpha}(\gamma;\beta;0;Q)$ *and the inequality*

$$\|u;\gamma,\beta,0;Q\|_{2+\alpha} \le c\left\{\sum_{k=0}^{N}\prod_{\lambda=k}^{N}(1+\|\psi_\lambda\|_{C^{2+\alpha}(Q\cap(t=t_\lambda))})\right.$$
$$\times\Big(\big\|\varphi_{k-1};\tilde{\gamma};\tilde{\beta};0;Q\cap(t=t_{k-1})\big\|_{2+\alpha} + \big\|f;\gamma;\beta;\mu_0;Q^{(k-1)}\big\|_{\alpha}$$
$$+\big\|g;\gamma;\beta;\delta;Q^{(k-1)}\big\|_{1+\alpha}\Big) + \big\|\varphi_N;\tilde{\gamma};\tilde{\beta};0;Q\cap(t=t_N)\big\|_{2+\alpha}$$
$$\left.+\big\|f;\gamma;\beta;\mu_0;Q^{N}\big\|_{\alpha} + \big\|g;\gamma;\beta;\delta;Q^{(N)}\big\|_{1+\alpha}\right\} \tag{7}$$

is true.

To prove Theorem 1, we first establish the correct solvability of boundary-value problems with smooth coefficients. From the obtained set of solutions, we select a convergent sequence whose limit value is the required solution of problem (1)–(5).

2. Estimation of the Solutions of Boundary-Value Problems with Smooth Coefficients

Let

$$Q_m^{(k)} = Q^{(k)} \cap \{(t,x) \in Q^{(k)} : s_1(1,t) \ge m_1^{-1}, s_2(1,x) \ge m_2^{-1}, m = (m_1,m_2), m_i > 1, i \in \{1,2\}\}$$

be sequences of domains that converge to $Q^{(k)}$ as $m_i \to \infty$.

In the domain Q, we consider the problem of determination of the solutions of the following equation:

$$(L_1 u_m)(t,x) \equiv \left[\partial_t - \sum_{i,j=1}^{n} a_{ij}(t,x)\partial_{x_i}\partial_{x_j} + \sum_{i=1}^{n} a_i(t,x)\partial_{x_i} + a_0(t,x)\right] u_m(t,x) = f_m(t,x), \quad (8)$$

satisfying the followings conditions in the variable t:

$$\begin{aligned} &u_m(t_0+0,x) = \varphi_0^{(0)}(x), \\ &u_m(t_\lambda+0,x) - u_m(t_\lambda-0,x) = \psi_m^{(\lambda)}(x)u_m(t_\lambda-0,x) + \varphi_m^{(\lambda)}(x) \end{aligned} \quad (9)$$

and the boundary conditions

$$\begin{aligned} &\lim_{x\to z\in\partial D}(B_1u_m - g_m)(t,x) = \lim_{x\to z\in\partial D}\left[\sum_{i=1}^{n} h_i(t,x)\partial_{x_i}u_m + h_0(t,x)u_m - g_m(t,x)\right] \geq 0, \\ &\lim_{x\to z\in\partial D} u_m \geq 0, \qquad \lim_{x\to z\in\partial D}[u_m(B_1u_m - g_m)] = 0. \end{aligned} \quad (10)$$

Here, the coefficients a_{ij}, a_i, a_0, h_i, and h_0 and the functions f_m, $\varphi_m^{(k)}$, $\psi_m^{(\lambda)}$, and g_m are determined as follows: If $(t,x) \in Q_m^{(k)}$, then the coefficients a_{ij}, a_i, a_0, h_i, and h_0 and the functions f_m, $\varphi_m^{(k)}$, $\psi_m^{(\lambda)}$, and g_m coincide with A_{ij}, A_i, A_0, b_i, b_0 and f, φ_k, ψ_λ, g, respectively. Moreover, in the domains $Q^{(k)}\backslash Q_m^{(k)}$, they are continuous extensions of the coefficients A_{ij}, A_i, A_0, b_i, and b_0 and the functions f, φ_k, ψ_λ, and g from the domains $Q_m^{(k)}$ to the domains $Q^{(k)}\backslash Q_m^{(k)}$ with preservation of smoothness and the norm [10, p. 82]. The following theorem is true for problem (8)–(10):

Theorem 2. *Suppose that* $u_m(t,x)$ *is a classical solution of problem (8)–(10) in the domain* Q *and conditions* $1°$ *and* $2°$ *are satisfied. Then the following estimate holds for* $u_m(t,x)$:

$$\begin{aligned} |u_m(t,x)| \leq \sum_{k=1}^{N}\Bigg\{\prod_{\lambda=k}^{N}\left(1+\left\|\psi_m^{(\lambda)};Q\cap(t=t_\lambda)\right\|_0\right)\Big(\left\|\varphi_m^{(k-1)};Q\cap(t=t_{k-1})\right\|_0 \\ + \left\|f_m a_0^{-1};Q^{(k-1)}\right\|_0 + \left\|g_m h_0^{-1};Q^{(k-1)}\right\|_0\Big)\Bigg\} \\ + \left\|\varphi_m^{(N)};Q\cap(t=t_N)\right\|_0 + \left\|f_m a_0^{-1};Q^N\right\|_0 + \left\|g_m h_0^{-1};Q^{(N)}\right\|_0. \end{aligned} \quad (11)$$

Proof. The validity of estimate (11) is established by using the method as in the proof of Theorem 2.2 in [4, p. 25]. Let

$$\max_{\bar{Q}^{(k)}} u_m(t,x) = u_m(P_1) > 0.$$

If $P_1 \in Q^{(k)}$, then the sufficient conditions for the existence of maximum of a function of many variables at the point P_1 guarantee the validity of the relations

$$\partial_t u_m(P_1) \geq 0, \qquad \partial_{x_i} u_m(P_1) = 0, \qquad \sum_{i,j=1}^{n} a_{ij}(P_1)\partial_{x_i}\partial_{x_j} u_m(P_1) \leq 0. \tag{12}$$

Moreover, $u_m(t,x)$ satisfies Eq. (8).

The last inequality in (12) is true because the second derivatives $\partial_{y_i}\partial_{y_j} u_m$ in any direction

$$y_j = \sum_{i=1}^{n} \alpha_{ij} s_1(\beta_i^{(1)}, t^{(1)}) s_2(\beta_i^{(2)}, x^{(1)})(x_i - x_i^{(1)}), \qquad \det\|\alpha_{ij}\| \neq 0,$$

are nonpositive at the point of maximum. Therefore,

$$\sum_{i,j=1}^{n} a_{ij}(P_1)\partial_{x_i}\partial_{x_j} u_m(P_1) = \sum_{\ell,j=1}^{n} \Bigg\{ \sum_{i,k=1}^{n} s_1(\beta_i^{(1)}, t^{(1)}) s_1(\beta_k^{(1)}, t^{(1)})$$

$$\times s_2(\beta_i^{(2)}, x^{(1)}) s_2(\beta_k^{(2)}, x^{(1)}) \alpha_{k\ell}\alpha_{ij} \Bigg\} \partial_{y_j}\partial_{y_\ell} u_m(P_1).$$

$$= \sum_{\ell=1}^{n} \lambda_\ell \partial_{y_j}\partial_{y_\ell} u_m < 0.$$

Since $\lambda_1, \ldots, \lambda_n$ are characteristic numbers of the quadratic form, they are positive according to restriction (6). In view of relations (12) and Eq. (8) at the point P_1, we get

$$u_m(P_1) \leq f_m(P_1) a_0^{-1}(P_1).$$

If $P_1 \in [t_k, t_{k+1}) \times \partial D$, then condition (9) is satisfied. Since

$$\frac{du_m(P_1)}{d\mathbf{e}} \geq 0$$

on $\Gamma^{(2)}$, the equality

$$\lim_{x \to z \in \partial D} [u_m(B_1 u_m - g_m)] = 0$$

yields the following inequality:

$$u_m(P_1) \leq \sup_{\bar{Q}^{(k)}} (g_m h_0^{-1}). \tag{13}$$

Similarly, for the point of the least negative value of the function in the case where

$$\min_{\bar{Q}^{(k)}} u_m = u_m(P_2) < 0\,,\ P_2 \in Q^{(k)},$$

we find

$$u_m(P_2) \geq f_m(P_2) a_0^{-1}(P_2). \tag{14}$$

If $P_2 \in [t_k, t_{k+1}) \times \partial D$, then

$$\frac{du_m(P_2)}{d\mathbf{e}} \leq 0 \quad \text{ on } \Gamma^{(2)}.$$

By using the equality $\lim\limits_{x \to z \in \partial D} [u_m(B_1 u_m - g_m)] = 0$, we obtain

$$u_m(P_2) \geq \inf_{\bar{Q}^{(k)}} (g_m h_0^{-1}). \tag{15}$$

In the case where $P_1 \in \bar{D}$ or $P_2 \in \bar{D}$, we derive the following inequality from the initial condition (9):

$$|u_m| \leq \left\| \varphi_m^{(0)}; D \right\|_0. \tag{16}$$

By using inequalities (13)–(16) for $k = 0$, we get

$$\left\| u_m; Q^{(0)} \right\| \leq \left\| f_m a_0^{-1}; Q^{(0)} \right\|_0 + \left\| g_m h_0^{-1}; Q^{(0)} \right\|_0 + \left\| \varphi_m^{(0)}; D \right\|_0. \tag{17}$$

If $P_1 \in Q \bigcap (t = t_\lambda)$ or $P_2 \in Q \bigcap (t = t_\lambda)$, $\lambda \geq 1$, then, by virtue of condition (9), we obtain the following recurrence relations:

$$\left\| u_m; Q \bigcap (t = t_\lambda) \right\| \leq \left(1 + \left\| \psi_m^{(\lambda)}; Q \bigcap (t = t_\lambda) \right\|_0 \right) \left\| u_m; Q^{(\lambda - 1)} \right\|_0 + \left\| \varphi_m^{(\lambda)}; Q \bigcap (t = t_\lambda) \right\|_0. \tag{18}$$

Combining inequalities (13)–(18), we deduce inequality (11).

In the domains $Q^{(k)}$, $k \in \{0, 1, \ldots, N\}$, we consider the problem

$$(L_1 u_m)(t,x) = f_m(t,x), \qquad u_m(t_k + 0, x) = G_m^{(k)}(t_k, x),$$

$$\lim_{x \to z \in \partial D} [u_m(B_1 u_m - g_m)] = 0, \qquad \lim_{x \to z \in \partial D} (B_1 u_m - g_m)(t,x) \geq 0\,, \tag{19}$$

$$\lim_{x \to z \in \partial D} u_m(t,x) \geq 0, \qquad t \in [t_k, t_{k+1}),$$

where

$$G_m^{(0)}(t_0,x) = \varphi_m^{(0)}(x), \qquad x \in D,$$

$$G_m^{(\lambda)}(t_k,x) = (1+\psi_m^{(\lambda)}(x))u_m(t_\lambda - 0,x) + \varphi_m^{(\lambda)},$$

$$x \in Q \cap (t = t_\lambda), \qquad \lambda \in \{1,2,\ldots,N\}.$$

In the domains $Q^{(k)}$, the solution of the boundary-value problem (19) exists and is unique in the space $C^{2+\alpha}(Q^{(k)})$ [4, p. 90].

We now estimate the derivatives of the solutions $u_m(t,x)$. In the space $C^{\ell}(Q)$, we introduce a norm $\|u_m;\gamma;\beta;q;Q\|_\ell$ equivalent, for fixed m_1 and m_2, to the Hölder norm of the same form as $\|u_m;\gamma;\beta;q;Q\|_\ell$ but with $d_1(q^{(1)},t)$ and $d_2(q^{(2)},x)$ taken instead of the functions $s_1(q^{(1)},t)$ and $s_2(q^{(2)},x)$, respectively:

$$d_1(q^{(1)},t) = \begin{cases} \max\{s_1(q^{(1)},t), m_1^{-q^{(1)}}\}, & q^{(1)} \ge 0, \\ \min\{s_1(q^{(1)},t), m_1^{-q^{(1)}}\}, & q^{(1)} < 0, \end{cases}$$

$$d_2(q^{(2)},x) = \begin{cases} \max\{s_2(q^{(2)},x), m_2^{-q^{(2)}}\}, & q^{(2)} \ge 0, \\ \min\{s_2(q^{(2)},x), m_2^{-q^{(2)}}\}, & q^{(2)} < 0. \end{cases}$$

The following theorem is true:

Theorem 3. *Suppose that conditions 1° and 2° are satisfied. Then the following estimate is true for the solution $u_m(t,x)$ of problem (8)–(10):*

$$\begin{aligned} \|u_m;\gamma,\beta,0;Q\|_{2+\alpha} \le c\Bigg\{\sum_{k=1}^{N}\Bigg\{\prod_{\lambda=k}^{N}(1+\|\psi_\lambda\|_{C^{2+\alpha}(Q\cap(t=t_\lambda))}) \\ \times\Big(\|\varphi_{k-1};\tilde{\gamma};\tilde{\beta};0;Q\cap(t=t_{k-1})\|_{2+\alpha} + \|f;\gamma;\beta;\mu_0;Q^{(k-1)}\|_\alpha \\ + \|g;\gamma;\beta;\delta;Q^{(k-1)}\|_{1+\alpha}\Big)\Bigg\} + \|\varphi_N;\tilde{\gamma};\tilde{\beta};0;Q\cap(t=t_N)\|_{2+\alpha} \\ + \|f;\gamma;\beta;\mu_0;Q^{(N)}\|_\alpha + \|g;\gamma;\beta;\delta;Q^{(N)}\|_{1+\alpha}\Bigg\}. \end{aligned} \tag{20}$$

The constant c is independent of m.

Proof. By using the definition of the norm and the interpolation inequalities from [9, 11], we obtain

$$\|u_m;\gamma;\beta;0;Q^{(k)}\|_{2+\alpha} \le (1+\varepsilon^\alpha)\langle u_m;\gamma;\beta;0;Q^{(k)}\rangle_{2+\alpha} + c(\varepsilon)\|u_m;Q^{(k)}\|_0,$$

where ε is an arbitrary real number from the interval $(0,1)$. Hence, it suffices to estimate the seminorm $\langle u_m;\gamma;\beta;0;Q^{(k)}\rangle_{2+\alpha}$.

In view of the definition of seminorm, in the domains $Q^{(k)}$, one can find points P_1, P_2, and H_i such that one of the following inequalities is true:

$$\frac{1}{2}\left\|u_m;\gamma;\beta;0;Q^{(k)}\right\|_{2+\alpha} \le E_\mu, \qquad \mu\in\{1,2\}, \tag{21}$$

where

$$E_1 \equiv \sum_{2s+|r|=2}\left\{\sum_{\nu=1}^{n} d_1(2s\gamma^{(1)},t^{(2)})d_2(2s\gamma^{(2)},\tilde{x})\prod_{i=1}^{n} d_1(r_i(\gamma^{(1)}-\beta_i^{(1)}),t^{(2)})\right.$$

$$\times d_2(r_i(\gamma^{(2)}-\beta_i^{(2)}),\tilde{x})\left|\partial_t^s\partial_x^r u_m(P_2)-\partial_t^s\partial_x^r u_m(H_\nu)\right|$$

$$\left.\times\left|x_\nu^{(1)}-x_\nu^{(2)}\right|^{-\alpha/2} d_1(\alpha(\gamma^{(1)}-\beta_\nu^{(1)}),t^{(1)})d_2(\alpha(\gamma^{(2)}-\beta_\nu^{(2)}),\tilde{x})\right\},$$

$$E_2 \equiv d_1((2+\alpha)\gamma^{(1)},\tilde{t})d_2((2s+\alpha)\gamma^{(2)},x^{(1)})\prod_{i=1}^{n} d_1(-r_i\beta_i^{(1)},\tilde{t})$$

$$\times d_2(r_i(\gamma^{(2)}-\beta_i^{(2)}),x^{(1)})\left|t^{(1)}-t^{(2)}\right|^{-\alpha/2}$$

$$\times\left|\partial_t^s\partial_x^r u_m(P_1)-\partial_t^s\partial_x^r u_m(P_2)\right|, \qquad 2s+|r| = 2.$$

If

$$\left|x_\nu^{(1)}-x_\nu^{(2)}\right| \ge \frac{\varepsilon_1}{4}\frac{1}{n} d_1(\gamma^{(1)},\tilde{t})d_2(\gamma^{(2)}-\beta_\nu^{(2)},\tilde{x}) \equiv T_1,$$

ε_1 is an arbitrary real number from the interval $(0,1)$, then

$$E_1 \le 2\varepsilon_1^{-\alpha}\left\|u_m;\gamma;\beta;0;Q^{(k)}\right\|_2.$$

If

$$\left|t^{(1)}-t^{(2)}\right| \ge \frac{\varepsilon_1^2}{16} d_1(2\gamma^{(1)},\tilde{t})d_2(2\gamma^{(2)},\tilde{x}) \equiv T_2,$$

then

$$E_2 \le 2\varepsilon_1^{-\alpha}\left\|u_m;\gamma;\beta;0;Q^{(k)}\right\|_2.$$

By using the interpolation inequalities for (16) and (17), we get

$$E_\mu \le \varepsilon^\alpha \left\| u_m;\gamma;\beta;0;Q^{(k)} \right\|_{2+\alpha} + c(\varepsilon)\left\| u_m;Q^{(k)} \right\|_0, \qquad \mu \in \{1,2\}. \tag{22}$$

Let

$$\left| x_j^{(1)} - x_j^{(2)} \right| \le T_2, \qquad \left| t^{(1)} - t^{(2)} \right| \le T_1 ,$$

$$d_1(\gamma^{(1)},\tilde{t}) \equiv d_1(\gamma^{(1)},t^{(1)}), \qquad d_2(\gamma^{(2)},\tilde{x}) \equiv d_2(\gamma^{(2)},x^{(1)}).$$

Assume that $|x_n - \xi_n| \le 2T_2$, $\xi \in \partial D$, or $|x - \xi| \le 2T_2 n$.

Consider a ball $\mathcal{K}(r,P)$ of radius r, $r \ge 4T_2 n$, containing the points P_1, H_i, and P_2 and centered at a point $P \in \Gamma$. In view of the restriction imposed on smoothness of the boundary ∂D, we can straighten $\partial D \cap \mathcal{K}(r,P)$ by using a one-to-one transformation $x = \psi(y)$ [11, p. 155]. As a result of this transformation, the domain $\Pi = Q^{(k)} \cap \mathcal{K}(r,P)$ turns into the domain Π_1 for the points of which, we have $y_n \ge 0$, $t \ge 0$.

Setting $u_m(t,x) = \omega_m(t,y)$, $P_1 = R_1$, $H_k = M_k$, $P_2 = R_2$, and $d_2(\gamma^{(2)},x^{(1)}) = p_2(\gamma^{(2)},y^{(1)})$ and denoting the coefficients of the differential expressions L_1 and B_1 under this transformation by k_{ij}, k_i, k_0, ℓ_i, and ℓ_0, we conclude that ω_m is a solution of the problem

$$\begin{aligned}\left[\partial_t - \sum_{i,j=1}^n k_{ij}(R_1)\partial_{y_i}\partial_{y_j}\right]\omega_m &= \sum_{i,j=1}^n [k_{ij}(t,y) - k_{ij}(R_1)]\partial_{y_i}\partial_{y_j}\omega_m \\ &\quad + \sum_{i=1}^n k_i(t,y)\partial_{y_i}\omega_m + k_0(t,y)\omega_m + F_m(t,\psi(y)) \\ &\equiv F_m^{(0)}(t,y),\end{aligned} \tag{23}$$

$$\omega_m(t_k + 0, y) = G_m^{(k)}(t_k,\psi(y)), \tag{24}$$

$$\omega_m\big|_{y_n=0} \ge 0,$$

$$\begin{aligned} B_2\omega_m\big|_{y_n=0} &\equiv \sum_{k=1}^n \ell_k(t,R_1)\partial_{y_k}\omega_m\Big|_{y_n=0} \\ &\ge \left[\sum_{k=1}^n \big(\ell_k(t,R_1) - \ell_k(t,y)\big)\partial_{y_k}\omega_m - \ell_0(t,y)\omega_m + g_m(t,\psi(y))e^{\lambda t}\right]\Bigg|_{y_n=0}, \\ &\equiv G(t,y)\big|_{y_n=0},\end{aligned}$$

$$\omega_m(B_2\omega_m - G)\big|_{y_n=0} = 0. \tag{25}$$

In problem (23)–(25), we perform the change of variables $\omega_m(t,y) = V_m(t,z)$, where

$$z_k = d_1(\beta_k^{(1)}, t^{(1)}) \times p_2(\beta_k^{(2)}, y^{(1)}) y_k, \quad k \in \{1, \ldots, n\}.$$

By Π_2 we denote the domain of definition of the functions $V_m(t,z)$. Then V_m is a solution of the problem

$$L_3 V_m \equiv \Bigg[\partial_t - \sum_{i,j=1}^{n} d_1(\beta_i^{(1)}, t^{(1)}) d_1(\beta_j^{(1)}, t^{(1)})$$

$$\times p_2(\beta_i^{(2)}, y^{(1)}) p_2(\beta_j^{(2)}, y^{(1)}) k_{ij}(R_1) \partial_{z_i} \partial_{z_j}\Bigg] V_m = F_m^{(0)}(t,Z),$$

$$V_m(t_k + 0, z) = G_m^{(k)}(Z) \equiv \Phi_m(Z),$$

$$V_m\big|_{z_n = m_2^{-1} p(\beta_n; R_1)} \geq 0,$$

$$B_3 V_m\big|_{z_n=0} \equiv \sum_{k=1}^{n} d_1(\beta_k^{(1)}, t^{(1)}) p_2(\beta_k^{(2)}, y^{(1)}) \ell_k(t, R_1) \partial_{z_k} V_m\big|_{z_n=0} \geq G(t,Z)\big|_{z_n=0},$$

$$V_m(B_3 V_m - G)\big|_{z_n=0} = 0,$$

where

$$Z = (d_1(-\beta_1^{(1)}, t^{(1)}) p_2(-\beta_1^{(2)}, y^{(1)}) z_1, \ldots, d_1(-\beta_n^{(1)}, t^{(1)}) p_2(-\beta_n^{(2)}, y^{(1)}) z_n).$$

We denote

$$z_i^{(1)} = d_1(\beta_i^{(1)}, t^{(1)}) p_2(\beta_i^{(2)}, y^{(1)}) y_i^{(1)},$$

$$\Pi_\mu^{(1)} = \left\{(t,z) \in \Pi_2 : \left|t - t^{(1)}\right| \leq \mu^2 T_2, \left|z_i - z_i^{(1)}\right| \leq \mu\sqrt{T_2}, i \in \{1, \ldots, n\}\right\}$$

and choose a three times differentiable function $\eta(t,z)$ satisfying the conditions

$$\eta(\tau,z) = \begin{cases} 1, & (t,z) \in \Pi_{1/2}^{(1)}, \quad 0 \leq \eta(t,z) \leq 1, \\ 0, & (t,z) \notin \Pi_{3/4}^{(1)}, \quad \left|\partial_t^k \partial_z^j \eta(t,z)\right| \leq c_{kj} d_1(-(2k+|j|)\gamma^{(1)}, t^{(1)}) p_2(-(2k+|j|)\gamma^{(2)}, y^{(2)}). \end{cases}$$

Then the function $W_m(t,z) = \eta(t,z) V_m(t,z)$ is a solution of the boundary-value problem

$$L_3 W_m = \sum_{i,j=1}^{n} d_1(\beta_i^{(1)}, t^{(1)}) d_1(\beta_j^{(1)}, t^{(1)}) p_2(\beta_i^{(2)}, y^{(1)}) p_2(\beta_j^{(2)}, y^{(1)}) k_{ij}(R_1)$$

$$\times[\partial_{z_i}\eta\partial_{z_j}V_m+\partial_{z_j}\eta\partial_{z_i}V_m]$$

$$+V_m\left[\sum_{i,j=1}^{n}d_1(\beta_i^{(1)},t^{(1)})d_1(\beta_j^{(1)},t^{(1)})p_2(\beta_i^{(2)},y^{(1)})\right.$$

$$\left.\times p_2(\beta_j^{(2)},y^{(1)})k_{ij}(R_1)\partial_{z_iz_j}\eta-\partial_t\eta\right]+F_m^{(0)}\eta\equiv F_m^{(1)}(t,z), \tag{26}$$

$$W_m(t_k+0,x)=\Phi_m(Z)\eta(t_k,z)\equiv\Phi_m^{(1)}(z), \tag{27}$$

$$W_m\big|_{z_n=0}\geq 0,$$

$$B_3W_m\big|_{z_n=0}\geq\left[\sum_{k=1}^{n}d_1(\beta_k^{(1)},t^{(1)})p_2(\beta_k^{(2)},y^{(1)})V_m\ell_k(t,R_1)\partial_{z_k}\eta-G(t,Z)\eta\right]\Bigg|_{z_n=0}\equiv G_1,$$

$$W_m(B_3W_m-G_1)\big|_{z_n=0}=0. \tag{28}$$

The following two cases are possible: either one can find points of the boundary $\Pi_2\cap\{z_n=0\}$ at which the condition

$$(B_3W_m-G_1)\big|_{z_n=0}=0 \tag{29}$$

is satisfied or these points do not exist, i.e.,

$$(B_3W_m-G_1)\big|_{z_n=0}>0.$$

Thus, it follows from the boundary condition (28) that

$$W_m\big|_{z_n=0}=0. \tag{30}$$

Assume that condition (29) is satisfied. In this case, we analyze problem (26), (27), (29). According to the imposed conditions, the coefficients of Eq. (26) and the boundary conditions (29) are bounded by constants independent of the point R_1. Thus, by using Theorem 6 in [4, p. 368], for arbitrary points $(M_1,M_2)\subset\Pi_{1/2}^{(1)}$, we obtain the inequality

$$d^{-\alpha}(M_1,M_2)\left|\partial_t^k\partial_z^jV_m(M_1)-\partial_t^k\partial_z^jV_m(M_2)\right|$$

$$\leq c\left(\left\|F_m^{(1)}\right\|_{C^{\alpha}(\Pi_{3/4}^{(1)})}+\left\|\Phi_m^{(1)}\right\|_{C^{2+\alpha}(\Pi_{3/4}^{(1)}\cap(t=t_k))}+\left\|G_1\right\|_{C^{1+\alpha}(\Pi_{3/4}^{(1)}\cap\{(t,z)\in\Pi_{3/4}^{(1)}|_{z_n=0}\})}\right), \tag{31}$$

where $2k+|j|=2$ and $d(M_1,M_2)$ is the parabolic distance between M_1 and M_2.

By using the properties of the function $\eta(t,z)$, we arrive at the estimates

$$\left\|F_m^{(1)}\right\|_{C^\alpha(\Pi_{3/4}^{(1)})} \le cd_1(-(2+\alpha)\gamma^{(1)},t^{(1)})p_2(-(2+\alpha)\gamma^{(2)},y^{(1)})$$
$$\times\left(\left\|F_m;\gamma;0,2\gamma;\Pi_{3/4}^{(1)}\right\|_\alpha+\left\|V_m;\Pi_{3/4}^{(1)}\right\|_0+\left\|V_m;\gamma;0,0;\Pi_{3/4}^{(1)}\right\|_2\right),$$

$$\left\|\Phi_m^{(1)}\right\|_{C^{2+\alpha}(\Pi_{3/4}^{(1)}\cap(t=0))} \le cd_1(-(2+\alpha)\gamma^{(1)},t^{(1)})p_2(-(2+\alpha)\gamma^{(2)},y^{(1)})$$
$$\times\left\|\Phi_m;\tilde\gamma;0;0;\Pi_{3/4}^{(1)}\cap(t=t_k)\right\|_{2+\alpha}, \tag{32}$$

$$\|G_1\|_{C^{1+\alpha}(\Pi_{3/4}^{(1)}\cap\{(t,z)\in\Pi_{3/4}^{(1)}|_{z_n=0}\})} \le cd_1(-(2+\alpha)\gamma^{(1)},t^{(1)})\ p_2(-(2+\alpha)\gamma^{(2)},y^{(1)})$$
$$\times\left(\left\|G;\gamma;0;\gamma;\Pi_{3/4}^{(1)}\right\|_{1+\alpha}+\left\|V_m;\gamma;0,0;\Pi_{3/4}^{(1)}\right\|_2+\left\|V_m;\Pi_{3/4}^{(1)}\right\|_0\right).$$

Substituting (32) in (31) and returning to the variables (t,y), we get

$$E_r \le c\Big(\|F_m;\gamma,\beta;2\gamma;\Pi_2\|_\alpha+\left\|\Phi_m^{(k)};\tilde\gamma,\tilde\beta;0;\Pi_2\cap(t=t_k)\right\|_{2+\alpha}$$
$$+\left\|G_1;\gamma,\beta;\gamma;\Pi_{3/4}^{(1)}\right\|_{1+\alpha}+c_1\|V_m;\gamma,\beta;0;\Pi_2\|_2+\|V_m;\Pi_2\|_0\Big),\qquad r\in\{1,2\}.$$

By using the definition of the space $H^{2+\alpha}(\gamma,\beta;0;Q)$ and conditions $1°$ and $2°$, we obtain

$$E_r \le c(n^2\rho^\alpha+\varepsilon^\alpha(n+2))\left\|u_m;\gamma;\beta;0;Q^{(k)}\right\|_{2+\alpha}$$
$$+c_1\Big(\left\|f_m;\gamma,\beta;2\gamma;Q^{(k)}\right\|_{2+\alpha}+\left\|G_m^{(k)};\tilde\gamma;\tilde\beta;0;Q\cap(t=t_k)\right\|_{2+\alpha}$$
$$+\|g_m;\gamma,\beta;\gamma;Q\|_{2+\alpha}+\|u_m;Q\|_0\Big), \tag{33}$$

where ε and ρ are arbitrary numbers, $\varepsilon\in(0,1)$, $\rho\in(0,1)$, and $r\in\{1,2\}$.

If condition (30) is satisfied, then we investigate problem (23), (24), (30). According to the imposed conditions, the coefficients of Eq. (23) are bounded by constants independent of the point R_1. Thus, by using Theorem 6.2 in [4, p. 368], for arbitrary points $(N_1,N_2)\subset\Pi_{1/2}^{(1)}$, we obtain

$$d^{-\alpha}(N_1,N_2)\left|\partial_t^j\partial_x^k v_m(N_1)-\partial_t^j\partial_x^k v_m(N_2)\right|$$
$$\le c\left(\left\|F_m^{(1)}\right\|_{C^\alpha(\Pi_{1/2}^{(1)})}+\left\|\Phi_m^{(1)}\right\|_{C^{2+\alpha}(\Pi_{3/4}^{(1)}\cap(t=t_k))}\right).$$

By virtue of the properties of the function $\eta(t,z)$, the definition of the space $H^{2+\alpha}(\gamma,\beta;0;Q)$, and conditions 1° and 2°, we find

$$E_r = c(n^2\rho^\alpha + \varepsilon^\alpha(n+2))\left\|u_m;\gamma;\beta;0;Q^{(k)}\right\|_{2+\alpha} + c_1\Big(\left\|f_m;\gamma,\beta;2\gamma;Q^{(k)}\right\|_{2+\alpha}$$

$$+\left\|G_m^{(k)};\tilde\gamma;\tilde\beta;0;Q\cap(t=t_k)\right\|_{2+\alpha} + \left\|u_m;Q\right\|_0\Big). \tag{34}$$

Let $|x-\xi|\ge 2T_2 n$. Consider a problem

$$\left[\partial_t - \sum_{i,j=1}^n a_{ij}(P)\partial_{x_i}\partial_{x_j}\right]u_m = \sum_{i,j=1}^n [a_{ij}(t,x)-a_{ij}(P_1)]\partial_{x_i}\partial_{x_j}u_m$$

$$+\sum_{i=1}^n (a_i(t,x)\partial_{x_i}u_m + a_0(t,x))u_m + f_m \equiv F_m^{(2)}(t,z), \tag{35}$$

$$u_m(t_k+0,x) = G_m^{(k)}(t_k,x). \tag{36}$$

Let $\Pi_1^{(2)}$ be a cube centered at the point P_1, $\Pi_1^{(2)}\in Q$, and let

$$\Pi_\rho^{(2)} = \{(t,x)\in Q^{(k)}:\left|t-t^{(1)}\right|\le 16^{-1}\mu^2 T, \left|x_i - x_i^{(1)}\right|\le 4\mu^{-1}T_2, i\in\{1,2,\ldots,n\}\}.$$

In problem (35), (36), we perform the change of variables

$$u_m(t,x)=\omega_m(t,y), \quad x_i = d_1(\beta_i^{(1)},t^{(1)})d_2(\beta_i^{(2)},x^{(1)})y_i, \quad i\in\{1,2,\ldots,n\}.$$

By $\Pi^{(3)}$ we denote the domain of definition of the functions $\omega_m(t,y)$. Then $\omega_m(t,y)$ is a solution of the problem

$$L_3\omega_m \equiv \Bigg[\partial_t - \sum_{i,j=1}^n a_{ij}(P_1)d_1(\beta_i^{(1)},t^{(1)})d_1(\beta_j^{(1)},t^{(1)})$$

$$\times d_2(\beta_i^{(2)},x^{(1)})d_2(\beta_j^{(2)},x^{(1)})\partial_{x_i}\partial_{x_j}\Bigg]\omega_m = F_m^{(2)}(t,Y),$$

$$\omega_m(t_k,y) = G_m^{(k)}(Y),$$

where

$$Y = (d_1(-\beta_1^{(1)},t^{(1)})d_2(-\beta_1^{(2)},x^{(1)})y_1,\ldots,d_1(-\beta_n^{(1)},t^{(1)})d_2(-\beta_n^{(2)},x^{(1)})y_n).$$

We denote

$$y_i^{(1)} = d_1(\beta_i^{(1)}, t^{(1)}) d_2(\beta_i^{(2)}, x^{(1)}) x_i^{(1)},$$

$$\Pi_\mu^{(3)} = \left\{(t,y) \in \Pi^{(3)} : \left|t - t^{(1)}\right| \le 16^{-1}\mu^2 T_1, \left|x_i^{(1)} - x_i\right| \le 4^{-1}\mu\sqrt{T_1}, i \in \{1,\ldots,n\}\right\}$$

and choose a three times differentiable function $\eta_1(t,y)$ satisfying the conditions

$$\eta_1(t,y) = \begin{cases} 1, & (t,y) \in \Pi_{1/2}^{(3)}, \quad 0 \le \eta_1(t,y) \le 1, \\ 0, & (t,y) \notin \Pi_{3/4}^{(3)}, \quad \left|\partial_t^k \partial_x^j \eta_1(t,y)\right| \le c_{kj} d_1(-(2k+|j|)\gamma^{(1)}, t^{(1)}) d_2(-(2k+|j|)\gamma^{(2)}, x^{(1)}). \end{cases}$$

Then the function $V_m^{(1)}(t,y) = \omega_m(t,y)\eta_1(t,y)$ is a solution of the Cauchy problem

$$\begin{aligned} L_3 V_m^{(1)} \equiv & \sum_{i,j=1}^n a_{ij}(P_1) d_1(\beta_i^{(1)}, t^{(1)}) d_1(\beta_j^{(1)}, t^{(1)}) d_2(\beta_i^{(2)}, x^{(1)}) d_2(\beta_j^{(2)}, x^{(1)}) \\ & \times [\partial_{y_i}\eta_1 \partial_{y_j}\omega_m + \partial_{y_j}\eta_1 \partial_{y_i}\omega_m] \\ & + \omega_m \left[\sum_{i,j=1}^n a_{ij}(P_1) d_1(\beta_i^{(1)}, t^{(1)}) d_1(\beta_j^{(1)}, t^{(1)}) \right. \\ & \left. \times d_2(\beta_i^{(2)}, x^{(1)}) d_2(\beta_j^{(2)}, x^{(1)}) \partial_{y_i}\partial_{y_j}\eta_1 - \partial_t \eta_1 \right] + F_m^{(2)}\eta_1 \equiv F_m^{(3)}, \end{aligned}$$

$$V_m^{(1)}(0,y) = G_m^{(k)}\eta_1(t_k, y) = \varphi_m^{(2)}.$$

According to Theorem 5.1 in [4, p. 364], for any points $(M_1, M_2) \subset \Pi_{1/2}^{(3)}$, the following inequality is true:

$$\begin{aligned} & d^{-\alpha}(M_1, M_2)\left|\partial_t^k \partial_x^j \omega_m(M_1) - \partial_t^k \partial_x^j \omega_m(M_2)\right| \\ & \qquad \le c\left(\left\|F_m^{(3)}\right\|_{C^\alpha(V_{3/4}^{(2)})} + \left\|\varphi_m^{(2)}\right\|_{C^{2+\alpha}(V_{3/4}^{(1)} \cap (t=t_k))}\right), \qquad 2k + |j| = 2. \end{aligned}$$

By using the properties of the function $\eta_1(t,y)$, the definition of the space $H^{2+\alpha}(\gamma;\beta;0;Q)$, and conditions 1° and 2°, we arrive at inequality (34). In view of the expression for $G_m^{(k)}(t_k, x)$, we arrive at the estimates

$$\left\|G_m^{(0)};\gamma;\beta;0;Q \cap (t=t_0)\right\|_{2+\alpha} \le \left\|\varphi_m^{(0)};\tilde{\gamma};\tilde{\beta};0;D\right\|_{2+\alpha}, \tag{37}$$

$$\left\|G_m^{(k)};\gamma;\beta;0;Q^{(k)}\cap(t=t_k)\right\|_{2+\alpha} \leq c\left(1+\left\|\psi_m^{(k)}\right\|_{C^{2+\alpha}(Q\cap(t=t_k))}\right)$$

$$\times\left\|u_m;\gamma;\beta;0;Q^{(k-1)}\right\|_{2+\alpha}+\left\|\varphi_m^{(k)};\tilde{\gamma};\tilde{\beta};0;Q\cap(t=t_k)\right\|_{2+\alpha}, \qquad (38)$$

$$k\in\{1,2,\ldots,N\}.$$

Combining inequalities (21), (22), (33), (34), (37), and (38) and choosing sufficiently small ε, we deduce the inequalities

$$\|u_m;\gamma;\beta;0;Q\|_{2+\alpha} \leq c\left\{\sum_{k=1}^{N}\left[\prod_{\lambda=k}^{N}\left(1+\left\|\psi_m^{(\lambda)}\right\|_{C^{2+\alpha}(Q\cap(t=t_\lambda))}\right)\right.\right.$$

$$\times\left(\left\|\varphi_m^{(k-1)};\tilde{\gamma};\tilde{\beta};0;Q\cap(t=t_{k-1})\right\|_{2+\alpha}+\left\|f_m;\gamma;\beta;\mu_0;Q^{(k-1)}\right\|_{\alpha}\right.$$

$$\left.\left.+\left\|g_m;\gamma;\beta;\delta;Q^{(k-1)}\right\|_{1+\alpha}\right)\right]+\left\|\varphi_m^{(N)};\tilde{\gamma};\tilde{\beta};0;Q\cap(t=t_N)\right\|_{2+\alpha}$$

$$\left.+\left\|f_m;\gamma;\beta;\mu_0;Q^{(N)}\right\|_{\alpha}+\left\|g_m;\gamma;\beta;\delta;Q^{(N)}\right\|_{1+\alpha}\right\}. \qquad (39)$$

Since

$$\left\|f_m;\gamma;\beta;\mu_0;Q^{(k)}\right\|_{\alpha} \leq c\left\|f;\gamma;\beta;\mu_0;Q^{(k)}\right\|_{\alpha},$$

$$\left\|\varphi_m^{(k)};\tilde{\gamma};\tilde{\beta};0;Q\cap(t=t_k)\right\|_{2+\alpha} \leq c\left\|\varphi_k;\tilde{\gamma};\tilde{\beta};0;Q\cap(t=t_k)\right\|_{2+\alpha}, \qquad (40)$$

$$\left\|g_m;\gamma;\beta;\delta;Q^{(k)}\right\|_{1+\alpha} \leq c\left\|g;\gamma;\beta;\delta;Q^{(k)}\right\|_{1+\alpha},$$

we substitute (40) in (39) and obtain inequality (20).

Proof of Theorem 1. The right-hand side of inequality (36) is independent of m_1 and m_2 and the sequences

$$\{u_m^{(0)}\} \equiv \{u_m(P)\}, \qquad P(t,x)\in Q^{(k)},$$

$$\{u_m^{(1)}\} \equiv \{d_1(\gamma^{(1)}-\beta_i^{(1)},t)\,d_2(\gamma^{(2)}-\beta_i^{(2)},x)\,\partial_{x_i}u_m\},$$

$$\{u_m^{(2)}\} \equiv \{d_1(2\gamma^{(1)},t)\,d_2(2\gamma^{(2)},x)\,\partial_t u_m\},$$

$$\{u_m^{(3)}\} = \{d_1(\gamma^{(1)}-\beta_i^{(1)},t)\,d_1(\gamma^{(1)}-\beta_j^{(1)},t)\,d_2(\gamma^{(2)}-\beta_i^{(2)},x)\,d_2(\gamma^{(2)}-\beta_j^{(2)},x)\,\partial_{x_ix_j}u_m\}$$

are uniformly bounded and equicontinuous in the domain $Q^{(k)}$. By the Arzelà theorem, there exist subsequences $\{u_{m(\ell)}^{(\mu)}\}$ uniformly convergent in $Q^{(k)}$ to $\{u^{(\mu)}\}$, $\mu \in \{0,1,2,3\}$. Passing to the limit as $m(\ell) \to \infty$ in problem (8)–(10), we conclude that $u(t,x) = u_0^{(0)}$ is a unique solution of problem (1)–(3), $u \in H^{2+\alpha}(\gamma;\beta;0;Q)$, and estimate (7) is true.

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