DOI 10.1007/s10958-021-05208-6
Journal of Mathematical Sciences, Vol. 253, No. 1, February, 2021

LOCALIZATION PROPERTY FOR REGULAR SOLUTIONS OF THE CAUCHY PROBLEM FOR A FRACTAL EQUATION OF THE INTEGRAL FORM

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UDC 517.956

We consider a fractal equation of the integral form with Bessel fractional integrodifferential operator and a positive parameter. In a part of the initial hyperplane, where the limit value has good properties, we establish the property of local strengthening of the convergence of regular solutions with generalized limit values.

Keywords: Cauchy problem, generalized limit value, localization property of the solution, Bessel fractional integrodifferential operator.

Introduction

In view of the nonlocality of fractional derivatives, numerous dynamical systems are described more exactly due to the application of differential equations of fractional orders. Various natural systems in different fields of science, such as the viscoelasticity, electric circuits, nonlinear oscillations caused by earthquakes, or diffusion processes in fractal media exhibit an intermediate behavior that can be modeled solely with the help of differential equations of fractional orders. Therefore, these equations form an important alternative to differential equations of integer orders. At present, they attract significant attention of the researchers and form a subject of active debates and discussions at various scientific meetings and conferences (see, e.g., [9, 14, 15, 17, 18]).

Consider a fractal equation of the integral form

$$\partial_t u(t,x) + \int_{-\infty}^{\gamma} ((a\mathcal{E} - \Delta_x)^{\tau/2} u)(t,x)\, d\tau = 0, \qquad (t,x) \in \Pi := (0,+\infty) \times \mathbb{R}^n, \tag{1}$$

where $\gamma > 0$, $(a\mathcal{E} - \Delta_x)^{\tau/2}$ is the Bessel operator of fractional integrodifferentiation with a parameter $a > 1$ [10, 13]. Moreover, the values of a and γ are such that the function $\tilde{\Omega}_\gamma(\cdot) := \tilde{P}_\gamma(\cdot) - \tilde{P}_\gamma(0)$ is convex on the set $[0,+\infty)$. Here, $\mathcal{E}$ is the identity operator, $\Delta_x := \partial_{x_1}^2 + \ldots + \partial_{x_n}^2$ is the Laplace operator, and

$$\tilde{P}_\gamma(\zeta) := \frac{(a+\zeta^2)^{\gamma/2}}{\ln (a+\zeta^2)^{1/2}}, \qquad \zeta \in \mathbb{R}.$$

In the Schwartz space S of rapidly decreasing functions [19], Eq. (1) is equivalent to the pseudodifferential equation [7]

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Translated from Matematychni Metody ta Fizyko-Mekhanichni Polya, Vol. 61, No. 2, pp. 7–17, April–June, 2018. Original article submitted April 16, 2018.

$$\partial_t u(t,x) + F^{-1}[P_\gamma(\xi)F[u](t,\xi)](t,x) = 0, \qquad (t,x) \in \Pi,$$

whose structure is close to the structure of the classical heat-conduction equation. Here, F is the Fourier transformation operator, $P_\gamma(\cdot) := \tilde{P}_\gamma(\|\cdot\|)$ denotes pseudodifferentiation, $\|\cdot\| := (\cdot,\cdot)^{1/2}$ is the Euclidean norm, and $(\cdot,\cdot)$ is the scalar product in $\mathbb{R}^n$.

In [7], the author established the absolute solvability of the Cauchy problem for Eq. (1) in special spaces W_β^Ω obtained as certain generalization of the well-known Gelfand, Shilov, and Gurevich spaces S and W [2, 4]. Actually, in the cited work, one can find the description of the entire set of initial data for which this problem is correctly solvable. In this case, its solution is regular and has the same property of smoothness and the same behavior in the neighborhoods of infinitely remote space points as the fundamental solution of the Cauchy problem.

It is worth noting that this set of initial data forms a sufficiently broad class some elements of which are generalized functions. Therefore, the initial condition of this Cauchy problem was formulated with a somewhat weakened notion of the limit. However, it is known from the classical theory of Cauchy problem for parabolic equations (and, in particular, for the heat-conduction equation) that the classical solutions $u(t,\cdot)$ of these equations with continuous limit values $f(\cdot)$ on the initial hyperplane $t=0$ approach $f(\cdot)$ as $t \to +0$ uniformly in the space variable on every compact part of this hyperplane (see [12, 14]). Thus, it is natural to ask whether the property of convergence in the initial condition of the Cauchy problem for Eq. (1) considered in [7] can be locally strengthened in the case where the initial generalized function f has "good" local properties. In other words, we arrive at the problem of localization principle.

Note that a similar problem was first considered in the theory of trigonometric series (see [11]). The investigations aimed at getting the localization principle were carried out for the solutions of partial differential equations parabolic in Petrovskii's and Shilov's sense in [3, 8, 16] and for the pseudodifferential equations with entire analytic symbols in [6].

The present work is devoted to the analysis of the property of localization of the regular solutions of Eq. (1) with generalized limit values in the form of Gurevich-type generalized functions. We preliminarily improve the estimate for the fundamental solution of the Cauchy problem with respect to the time variable and perform an extension of the space of test functions by finite functions required for the correct definition of the equality of generalized functions on a set. The accumulated results are illustrated on a model example.

1. Preliminary Information

Let $\mathbb{N}$ be the set of all natural numbers, let $\mathbb{R}^n$ and $\mathbb{C}^n$ be, respectively, the real and complex spaces of dimension n, and let $\mathbb{Z}_+^n$ be the set of all n-dimensional multiindices. We use the following notation: i is the imaginary unit;

$$z^\ell := z_1^{\ell_1} \ldots z_n^{\ell_n}, \quad |\ell| := \ell_1 + \ldots + \ell_n \quad \text{for} \quad z := (z_1,\ldots,z_n) \in \mathbb{C}^n \quad \text{and} \quad \ell := (\ell_1,\ldots,\ell_n) \in \mathbb{Z}_+^n;$$

$C^\infty(\mathbb{R}^n)$ is the class of all functions infinitely differentiable on $\mathbb{R}^n$; S is the Schwartz space of rapidly decreasing functions; $L_2(\mathbb{R}^n)$ is the Lebesgue space of functions square summable on $\mathbb{R}^n$, and $\mu(\cdot)$ is a function continuously increasing on $[0,+\infty)$ and such that

$$\lim_{\tau \to +\infty} \mu(\tau) = +\infty \qquad \text{and} \qquad \mu(0) = 0.$$

We put

$$\tilde{M}(\tau) := \int_0^\tau \mu(\xi)\,d\xi\,, \ \ \tau \ge 0\,.$$

In the set $[0,+\infty)$, the function $\tilde{M}(\cdot)$ is increasing, differentiable, and convex. Moreover, $\tilde{M}(0)=0$ and

$$\lim_{\tau\to+\infty} \tilde{M}(\tau) = +\infty\,.$$

Parallel with $\tilde{M}(\cdot)$, we consider a similar function $\tilde{\Omega}(\cdot)$ constructed according to the function $\omega(\cdot)$, which has the same properties as $\mu(\cdot)$.

Let $M(x) := \tilde{M}(\|x\|)$ and $\Omega(x) := \tilde{\Omega}(\|x\|)$, $x \in \mathbb{R}^n$. For $\alpha > 0$ and $\beta > 0$, we set

$$S_\alpha^\beta = \{\varphi \in C^\infty(\mathbb{R}^n) \,|\, \exists c>0, \exists A>0, \exists \delta>0 \ \forall k \in \mathbb{Z}_+^n, \forall x \in \mathbb{R}^n : \left|\partial_x^k \varphi(x)\right| \le cA^{|k|}|k|^{\beta|k|}e^{-\delta\|x\|^{1/\alpha}}\},$$

$$W_M^\beta = \{\varphi \in C^\infty(\mathbb{R}^n) \,|\, \exists c>0, \exists A>0, \exists \delta>0 \ \forall k \in \mathbb{Z}_+^n, \forall x \in \mathbb{R}^n : \left|\partial_x^k \varphi(x)\right| \le cA^{|k|}|k|^{\beta|k|}e^{-M(\delta x)}\}.$$

The space W_α^Ω is defined as the set of all elements from the class $C^\infty(\mathbb{R}^n)$ admitting analytic extensions in $\mathbb{C}^n$ to entire functions such that

$$\exists c>0, \quad \exists \lambda>0, \quad \exists \delta>0 \quad \forall z = x+iy \in \mathbb{C}^n : |\varphi(z)| \le ce^{-\delta\|x\|^{1/\alpha}+\Omega(\lambda y)}.$$

In [2, 7], these spaces are endowed with related topologies. With the indicated topologies, these spaces turn into unions of complete perfect countably normed spaces. In particular, a sequence of elements $\{\varphi_\nu(\cdot), \nu \in \mathbb{N}\} \subset S_\alpha^\beta$ in the space S_α^β

1) is bounded if

$$\exists c>0, \quad \exists A>0, \quad \exists \delta>0 \quad \forall k \in \mathbb{Z}_+^n, \quad \forall x \in \mathbb{R}^n, \forall \nu \in \mathbb{N} : \left|\partial_x^k \varphi_\nu(x)\right| \le cA^{|k|}|k|^{\beta|k|}e^{-\delta\|x\|^{1/\alpha}}\,;$$

2) converges to zero (we denote $\varphi_\nu \xrightarrow[\nu\to+\infty]{S_\alpha^\beta} 0$) if it is bounded in this space and regularly convergent, i.e.,

$$\forall k \in \mathbb{Z}_+^n, \qquad \forall \mathcal{K} \subset \mathbb{R}^n : \partial_x^k \varphi_\nu(x) \underset{\nu\to+\infty}{\overset{x\in\mathcal{K}}{\rightrightarrows}} 0$$

(here, we speak about uniform convergence with respect to x on every compact set $\mathcal{K}$).

If $M(\cdot)$ and $\bar{M}(\cdot)$ are functions constructed according to the corresponding $\mu(\cdot)$ and $\bar{\mu}(\cdot)$ that are dual in Young's sense (see the definition in [4]), then the following topological equalities are true:

$$F[W_\alpha^M] = W_{\bar{M}}^\alpha, \qquad F[S_\alpha^\beta] = S_\beta^\alpha. \tag{2}$$

Here, the operator F is continuous in each of these spaces.

The results obtained in [7] immediately imply that the function $g(\cdot)\in C^{\infty}(\mathbb{R}^n)$ is a multiplier in the space $\Phi\in\{S_{\alpha}^{\beta},W_{\Omega}^{\beta}\}$ only if

$$\forall\delta>0,\quad \exists c_{\delta}>0,\quad \exists A_{\delta}>0\quad \forall k\in\mathbb{Z}_{+}^{n},\quad \forall x\in\mathbb{R}^n: \left|\partial_x^k g(x)\right| \le c_{\delta}A_{\delta}^{|k|}|k|^{\beta|k|}E_{\delta}(x), \tag{3}$$

where

$$E_{\delta}(x)=\begin{cases} e^{\delta\|x\|^{1/\alpha}}, & \Phi=S_{\alpha}^{\beta},\\ e^{\Omega(\delta x)}, & \Phi=W_{\Omega}^{\beta}.\end{cases}$$

Thus, in view of (2) and the equality

$$F[f*\varphi](\cdot)=F[f](\cdot)F[\varphi](\cdot),\qquad \forall\varphi\in F[\Phi],$$

which is true for the convolvers $f\in F[\Phi']$ [1], we conclude that f from $F[\Phi']$ is a convolver in the space $F[\Phi]$ iff its Fourier transform $F[f](\cdot)$ is a multiplier in Φ, i.e., iff $F[f](\cdot)$ satisfies the corresponding condition (3). Here, Φ' is the space topologically dual to Φ.

Let $\tilde{M}_{\gamma}(\cdot)$ be a function dual in Young's sense to $\tilde{\Omega}_{\gamma}(\cdot)$ and let

$$\Omega_{\gamma}(x):=\tilde{\Omega}_{\gamma}(\|x\|)\qquad\text{and}\qquad M_{\gamma}(x):=\tilde{M}_{\gamma}(\|x\|),\quad x\in\mathbb{R}^n.$$

For Eq. (1), we impose the following initial condition:

$$u(t,\cdot)\xrightarrow[t\to+0]{(W_{\beta}^{M_{\gamma}})'} f,\qquad f\in(W_{\beta}^{M_{\gamma}})', \tag{4}$$

(in this case, we speak about weak convergence in the space $(W_{\beta}^{M_{\gamma}})'$).

The following assertion is true [7]:

In order that the Cauchy problem (1), (4) be correctly solvable and its solution $u(t,\cdot)$ *belong to the space* $W_{\beta}^{M_{\gamma}}$, $\beta\ge 1$, *it is necessary and sufficient that the initial generalized function* f *from* $(W_{\beta}^{M_{\gamma}})'$ *be a convolver in the space* $W_{\beta}^{M_{\gamma}}$. *In this case, the equality*

$$u(t,x)=(f*G)(t,x)\equiv\langle f,G(t,x-\cdot)\rangle,\qquad (t,x)\in\Pi, \tag{5}$$

holds and the derivatives satisfy the following relations on the set Π:

$$\partial_t u(t,x)=\langle f,\partial_t G(t,x-\cdot)\rangle,\qquad \partial_x^k u(t,x)=\langle f,\partial_x^k G(t,x-\cdot)\rangle,\qquad k\in\mathbb{Z}_{+}^{n}.$$

Here, G is the fundamental solution of problem (1), (4), and the action of a generalized function upon a test function is denoted by $\langle \cdot , \cdot \rangle$.

In [7], it was established that

$$G(t,\cdot) = F^{-1}[\theta_t(\xi)](t,\cdot), \quad t > 0,$$

where

$$\theta_t(\xi) := e^{-tP_\gamma(\xi)}, \qquad (t,\xi) \in \Pi,$$

and, moreover, $\theta_t(\cdot)$ belongs to the space $W^1_{\Omega_\gamma}$ for any fixed $t > 0$.

Thus, in view of relation (2), we arrive at the following assertion:

Corollary 1. *For any fixed $t > 0$, the fundamental solution $G(t,\cdot)$ is an element of the space $W_1^{M_\gamma}$.*

To study the properties of the solution $u(t,\cdot)$ of the Cauchy problem (1), (4) with respect to the variable t, it is first necessary to clarify the properties of the fundamental solution $G(t,\cdot)$ with respect to this variable. Therefore, in the first turn, we get the exact estimates for the derivatives $\partial_\xi^k \theta_t(\xi)$ by separating, in all cases, the dependence on t.

To simplify calculations, we consider the case where $n = 1$ (the general case $n \in \mathbb{N}$ is studied by induction). According to the Faà di Bruno formula [5], we obtain

$$\left|\partial_\xi^k \theta_t(\xi)\right| \le \sum_p^k \frac{k!}{i!j!\ldots h!} t^p \theta_t(\xi) \left|\frac{dP_\gamma(\xi)}{1!d\xi}\right|^i \left|\frac{d^2P_\gamma(\xi)}{2!d\xi^2}\right|^j \ldots \left|\frac{d^\ell P_\gamma(\xi)}{\ell!d\xi^\ell}\right|^h, \qquad (t,\xi) \in \Pi,$$

where the operation of summation is carried out over all integer nonnegative solutions of the equation $k = i + 2j + \ldots + \ell h$, and the number $p = i + j + \ldots + h$. By using the following estimates from [7]:

$$\left|\frac{d^m P_\gamma(\xi)}{m!d\xi^m}\right| \le cA^m \frac{(a+\xi^2)^{(\gamma-m)/2}}{\ln(a+\xi^2)^{1/2}},$$

$$\frac{p!}{i!j!\ldots h!} \le 2^k, \qquad \sum_p^k 1 \le (2e)^k,$$

we find

$$\left|\partial_\xi^k \theta_t(\xi)\right| \le A^k k! \theta_t(\xi) \sum_p^k \frac{(ctP_\gamma(\xi))^p}{i!j!\ldots h!} (a+\xi^2)^{-k/2}$$

$$\le A^k k! t^{k/\gamma} \theta_t(\xi) \sum_p^k \frac{c^p (tP_\gamma(\xi))^{p-k/\gamma}}{i!j!\ldots h!}$$

$$\le c_1 A_1^k k! t^{k/\gamma} \theta_{t/2}(\xi) \sum_p^k \frac{\sup_{\tau>0} \{\tau^{p-k/\gamma} e^{-\tau}\}}{i!\, j! \ldots h!}$$

$$\le c_2 A_2^k k! t^{k/\gamma} \theta_{t/2}(\xi), \qquad (t,\xi) \in \Pi,$$

where A, c, A_j, and c_j are positive constants independent of t, x, and k.

Hence, in view of the relations

$$\forall \alpha > 0 \qquad \exists c_\alpha > 0 \qquad \forall \xi \in \mathbb{R}^n : P_\gamma(\xi) \ge c_\alpha (a + \|\xi\|^2)^{(\gamma-\alpha)/2}, \tag{6}$$

for $\{k, q\} \subset \mathbb{Z}_+^n \Omega$ and $(t,x) \in \Pi$, we get

$$\left| x^k \partial_x^q G(t,x) \right| = (2\pi)^{-n} \left| \int_{\mathbb{R}^n} e^{-i(x,\xi)} \partial_\xi^k (\xi^q \theta_t(\xi)) \, d\xi \right|$$

$$\le \sum_{\ell=0}^k C_k^\ell \int_{\mathbb{R}^n} \left| \partial_\xi^\ell \xi^q \right| \left| \partial_\xi^{k-\ell} \theta_t(\xi) \right| d\xi$$

$$\le c_2 \sum_{\ell=0}^k C_k^\ell A_2^{|k-\ell|} (k-\ell)! t^{|k-\ell|/\gamma} \frac{q!}{(q-\ell)!}$$

$$\times \int_{\mathbb{R}^n} \|\xi\|^{|q-\ell|} e^{-\frac{c_\alpha}{2} t \|\xi\|^{\gamma-\alpha}} \, d\xi$$

$$\le c_2 \sum_{\ell=0}^k C_k^\ell A_2^{|k-\ell|} (4 c_\alpha^{-1})^{\frac{|q-\ell|}{\gamma-\alpha}} t^{\frac{|k-\ell|}{\gamma} - \frac{|q-\ell|}{\gamma-\alpha}} \frac{(k-\ell)! q!}{(q-\ell)!}$$

$$\times \sup_{\tau>0} \left\{ \tau^{\frac{|q-\ell|}{\gamma-\alpha}} e^{-\tau} \right\} \int_{\mathbb{R}^n} e^{-\frac{c_\alpha}{4} t \|\xi\|^{\gamma-\alpha}} \, d\xi$$

$$\le c_\alpha A^{|k|} B_\alpha^{|q|} k! |q|^{\frac{|q|}{\gamma-\alpha}} t^{\frac{|k|}{\gamma} - \frac{n+|q|\gamma_0}{\gamma-\alpha}} \qquad \forall \alpha \in (0, \gamma).$$

Here, the estimated quantities c_α, A, and B_α are positive and independent of t, x, k, and q. In this case, A is also independent of α and

$$\gamma_0 = \begin{cases} 1, & t \in (0,1), \\ \gamma_0 = 1 - \alpha/\gamma, & t \ge 1. \end{cases}$$

Hence, the following proposition is true:

Lemma 1. *The function* $G(t,\cdot)$, $t>0$, *is infinitely differentiable on the set* $\mathbb{R}^n$ *and, in addition,*

$$\exists\delta>0 \quad \forall\alpha\in(0,\gamma) \quad \exists c_\alpha>0, \quad \exists B_\alpha>0, \quad \forall q\in\mathbb{Z}_+^n,$$

$$\forall(t,x)\in\Pi: \left|\partial_x^q G(t,x)\right| \le c_\alpha B_\alpha^{|q|} |q|^{\frac{|q|}{\gamma-\alpha}} t^{-\frac{n+|q|\gamma_0}{\gamma-\alpha}} e^{-\delta\|x\|/t^\gamma}.$$

Corollary 2. *The fundamental solution* $G(t,\cdot)$ *of the Cauchy problem for Eq. (1) is an element of the space* $S_1^{1/(\gamma-\alpha)}$ *for all* $\alpha\in(0,\gamma)$ *and any fixed* $t>0$.

2. Localization Principle

As $t\to+0$, the solution of the Cauchy problem (1), (4) approaches a generalized function f in a sense of weak convergence. However, it is possible that, in a certain domain $Q\subset\mathbb{R}^n$, f coincides with a smooth function. Is it true that a local strengthening of convergence of the indicated solution occurs in the indicated case? We now try to answer this question.

Note that the notion of equality of generalized functions on the set $Q\subset\mathbb{R}^n$ requires the presence of finite functions in the corresponding space of test functions [2]:

The elements f *and* g *from the space* Φ' *are equal on the set* Q *provided that*

$$\langle f-g,\varphi(\cdot)\rangle = 0 \qquad \forall\varphi(\cdot)\in\Phi, \qquad \operatorname{supp}\varphi\subset Q.$$

By definition, the space $W_\beta^{M_\gamma}$ is formed solely by entire analytic functions. Therefore, it is first necessary to restrict, in a proper way, the corresponding space $(W_\beta^{M_\gamma})'$ of initial data of the Cauchy problem (1), (4) by complementing the space $W_\beta^{M_\gamma}$ by smooth finite functions.

Estimate (6) guarantees the validity of the embedding

$$W_{\Omega_\gamma}^\beta \subset S_{1/(\gamma-\alpha)}^\beta$$

for $\beta>0$ and $\alpha\in(0,\gamma)$. Hence, in view of (2), we arrive at the following chain of continuous embeddings:

$$W_\beta^{M_\gamma} \subset S_\beta^{1/(\gamma-\alpha)} \subset S_\beta^r \subset S \subset L_2(\mathbb{R}^n) \subset S' \subset (S_\beta^r)' \subset (S_\beta^{1/(\gamma-\alpha)})' \subset (W_\beta^{M_\gamma})'$$

for all $\beta>0$, $\alpha\in(0,\gamma)$, and $(\gamma-\alpha)^{-1}\le r$. In view of the presence of finite functions in the space S_α^β for $\beta>1$ [2], this yields the required restrictions of the space of initial data:

$$S' \subset (S_\beta^r)' \subset (S_\beta^{1/(\gamma-\alpha)})' \subset (W_\beta^{M_\gamma})', \qquad \beta>0, \quad 1<(\gamma-\alpha)^{-1}\le r.$$

Two following assertions are true:

Theorem 1. *Let* $\beta \ge 1$ *and let the parameters* $\alpha \in (0,\gamma)$ *and* $r>0$ *be such that*

$$1 < (\gamma-\alpha)^{-1} \le \frac{r\gamma}{\gamma+1}. \tag{7}$$

If a generalized function $f \in (S_\beta^r)'$ *is equal to zero on a set* $Q \subset \mathbb{R}^n$, *then the corresponding function* $u(t,\cdot)$, $t>0$, *given by equality (5) converges to zero together with all its derivatives* $\partial_x^k u(t,x)$ *as* $t \to +0$ *uniformly in the variable* x *on every compact set* $\mathcal{K} \subset Q$.

Proof. Let $\mathcal{K} \subset \mathcal{K}_1 \subset Q$, where $\mathcal{K}_1$ is a compact set from $\mathbb{R}^n$ such that

$$\forall x \in \mathcal{K}, \qquad \forall \xi \in \mathbb{R}^n \setminus \mathcal{K}_1 : \|x-\xi\| \ge b > 0. \tag{8}$$

Consider a finite function $\eta(\cdot) \in S_\beta^{1/(\gamma-\alpha)}$ whose support is located in Q such that $\eta(\xi)=1$, $\xi \in \mathcal{K}_1$. We set $\nu(\cdot) := 1-\eta(\cdot)$. Since the operations of multiplication and ordinary shift of the argument are defined in the spaces S_α^β, Corollary 2 implies that

$$\{\eta(\cdot)\partial_x^k G(t,x-\cdot), \nu(\cdot)\partial_x^k G(t,x-\cdot)\} \subset S_\beta^{1/(\gamma-\alpha)}, \qquad (t,x) \in \Pi.$$

Thus, for all $k \in \mathbb{Z}_+^n$ and $(t,x) \in \Pi$, we arrive at the representation

$$\partial_x^k u(t,x) = \left\langle f, \eta(\cdot)\partial_x^k G(t,x-\cdot)\right\rangle + \left\langle f, \nu(\cdot)\partial_x^k G(t,x-\cdot)\right\rangle.$$

In view of the equality $f=0$ valid on Q and the relation

$$\operatorname{supp}(\eta(\cdot)\partial_x^k G(t,x-\cdot)) \subset Q,$$

this yields

$$\partial_x^k u(t,x) = t^{\alpha_1/(\gamma-\alpha)}\left\langle f, \omega_{t,x}^k(\cdot)\right\rangle, \qquad \alpha_1 > 0, \qquad (t,x) \in \Pi,$$

where $\omega_{t,x}^k(\cdot) := t^{-\alpha_1/(\gamma-\alpha)}\nu(\cdot)\partial_x^k G(t,x-\cdot)$.

Hence, in order to prove the theorem, it is sufficient to establish the uniform boundedness of the family $\omega_{t,x}^k(\cdot)$ in the space S_β^r with respect to the parameters $0<t\ll 1$ and $x \in \mathcal{K}$, i.e., to prove the following estimate:

$$\left|\partial_\xi^q \omega_{t,x}^k(\xi)\right| \le cA^{|q|}|q|^{r|q|}e^{-\delta\|\xi\|^{1/\beta}} \qquad \forall q \in \mathbb{Z}_+^n, \tag{9}$$

where the positive constants c, A, and δ are independent of t, x, ξ, and q. Since $\omega^k_{t,x}(\xi)=0$ for $\xi\in\mathcal{K}_1$, it is sufficient to establish estimate (9) only for $\xi\in\mathbb{R}^n\backslash\mathcal{K}_1$.

By using the Leibnitz formula of differentiation of the product of functions, assertion of Lemma 1, and the fact that $\eta(\cdot)$ belongs to $S_\beta^{1/(\gamma-\alpha)}$, we find

$$\left|\partial_\xi^q \omega^k_{t,x}(\xi)\right| \le t^{-\frac{\alpha_1}{\gamma-\alpha}}\left\{\left|\partial^{k+q}_{x-\xi}G(t,x-\xi)\right|\right.$$

$$\left.+\sum_{\ell=0}^{q} C_q^\ell \left|\partial_\xi^{q^{-1}}\eta(\xi)\right|\left|\partial^{\ell+k}_{x-\xi}G(t,x-\xi)\right|\right\}$$

$$\le c_\alpha t^{-\frac{\alpha_1+n+|k+q|}{\gamma-\alpha}} e^{-\delta\|x-\xi\|/t^\gamma}\left\{B_\alpha^{|k+q|}\|k+q\|^{\frac{|k+q|}{\gamma-\alpha}}\right.$$

$$\left.+\sum_{\ell=0}^{q} C_q^\ell A^{|q-\ell|}B_\alpha^{|k+q|}|q-\ell|^{\frac{|q-\ell|}{\gamma-\alpha}}|k+\ell|^{\frac{|k+\ell|}{\gamma-\alpha}}\right\}$$

$$\le cB^{|k+q|}|k+q|^{\frac{|k+q|}{\gamma-\alpha}} t^{-\frac{\alpha_1+n+|k+q|}{\gamma-\alpha}} e^{-\delta\|x-\xi\|/t^\gamma}$$

$$\le cB^{|k+q|}|k+q|^{\frac{|k+q|}{\gamma-\alpha}}\sup_{t\in(0,1)}\left\{t^{-\frac{\alpha_1+n+|k+q|}{\gamma-\alpha}}e^{-\frac{\delta b}{2t^\gamma}}\right\}e^{-\frac{\delta}{2}\|x-\xi\|}$$

$$\le c_1 B_1^{|k+q|}|k+q|^{\frac{|k+q|(1+\gamma)}{\gamma(\gamma-\alpha)}}\sup_{x\in K}\{e^{\delta_1\|x\|}\}e^{-\delta_0\|\xi\|},$$

$$0<t\ll 1,\qquad x\in\mathcal{K},\qquad \xi\in\mathbb{R}^n\backslash\mathcal{K}_1,\qquad \{k,q\}\subset\mathbb{Z}^n_+.$$

Thus, estimate (9) now becomes obvious. Theorem 1 is proved.

Theorem 2. *Let* $\gamma>1$, *let* $\alpha\in(\gamma-1,\gamma-1/\gamma]$, *and let* $r\ge\dfrac{\gamma+1}{\gamma(\gamma-\alpha)}$. *If a generalized function* $f\in(S_\beta^r)'$, $\beta\ge1$, *coincides, on a set* $Q\subset\mathbb{R}^n$, *with a function* $g(\cdot)$ *continuously differentiable on* Q *up to the order* $q_0\in\mathbb{Z}^n_+$, *inclusively, then the corresponding function* $u(t,\cdot)=\langle f,G(t,x-\cdot)\rangle$ *converges, together with all its derivatives* $\partial_x^k u(t,x)$, $k\le q_0$, *to* $\partial_x^k g(x)$ *as* $t\to+0$ *uniformly in the variable* x *on every compact set* $\mathcal{K}\subset Q$.

Proof. First, we note that the fact that α belongs to $\left(\gamma-1,\dfrac{\gamma-1}{\gamma}\right]$ is equivalent to the condition

$$1<(\gamma-\alpha)^{-1}\le\gamma$$

and

$$r \geq \frac{\gamma+1}{\gamma(\gamma-\alpha)} \Leftrightarrow (\gamma-\alpha)^{-1} \leq r\gamma \leq (\gamma+1).$$

Therefore, the conditions imposed on the parameters α and r in the analyzed theorem guarantee the validity of condition (7) from Theorem 1 for these parameters.

We now use the quantities $\mathcal{K}$, $\mathcal{K}_1$, $\eta(\cdot)$, and $\nu(\cdot)$ from the proof of the previous assertion.

It follows from the equalities $f-g=0$ on Q and $\nu f=0$ on $\mathcal{K}_1$, the assertion of Theorem 1, and the representation

$$\partial_x^k u(t,x) = \left\langle \eta(f-g), \partial_x^k G(t,x-\cdot)\right\rangle + \left\langle \nu f, \partial_x^k G(t,x-\cdot)\right\rangle + \left\langle \eta g, \partial_x^k G(t,x-\cdot)\right\rangle$$

that, in order to prove Theorem 2, it suffices to establish the following limit relation:

$$I_t^k(x) := \left\langle \eta g, \partial_x^k G(t,x-\cdot)\right\rangle \underset{t\to+0}{\overset{x\in\mathcal{K}}{\rightrightarrows}} \partial_x^k(\eta g)(x).$$

In view of the regularity of the functional ηg and the equality

$$\partial_x^k G(t,x-\xi) = (-1)^{|k|}\partial_\xi^k G(t,x-\xi),$$

as a result of integration by parts, we arrive at the representation

$$I_t^k(x) = \int_{\mathbb{R}^n} G(t,x-\xi)\partial_\xi^k(\eta g)(\xi)\,d\xi, \qquad k \leq q_0.$$

By using this representation and the equality

$$\int_{\mathbb{R}^n} G(t,x-\xi)\,d\xi = \theta_t(0), \qquad t>0,$$

for $(t,x)\in\Pi$ and $k\leq q_0$, we find

$$\left|I_t^k(x) - \partial_x^k(\eta g)(x)\right|$$

$$= \left|\int_{\mathbb{R}^n} G(t,x-\xi)(\partial_x^k(\eta g)(x) - \partial_\xi^k(\eta g)(\xi))\,d\xi - O(t)\partial_x^k(\eta g)(x)\right|$$

$$\leq \int_{\mathbb{R}^n} |G(t,\zeta)|\left|\partial_{x-\zeta}^k(\eta g)(x-\zeta) - \partial_x^k(\eta g)(x)\right| d\zeta$$

$$+|O(t)|\left|\partial_x^k(\eta g)(x)\right|,$$

where

$$O(t) := 1-\theta_t(0) \underset{t\to+0}{\to} 0.$$

The function $\partial_x^k(\eta g)(x)$ is continuous and $\operatorname{supp}(\partial_x^k(\eta g)) \subset Q$. Therefore,

(i) $\sup\limits_{x\in\mathcal{K}}\left|\partial_x^k(\eta g)(x)\right| = N_1 < +\infty$;

(ii) $\sup\limits_{x\in\mathcal{K},\,\zeta\in\mathbb{R}^n}\left|\partial_{x-\zeta}^k(\eta g)(x-\zeta)-\partial_x^k(\eta g)(x)\right| = N_2 < +\infty$;

(iii) $\forall\varepsilon>0\ \exists\delta_0>0\ \forall x\in\mathcal{K},\ \forall\zeta\in\mathbb{R}^n,\ ||x-\zeta-x|| = ||\zeta|| < \delta_0 : \left|\partial_{x-\zeta}^k(\eta g)(x-\zeta)-\partial_x^k(\eta g)(x)\right| < \varepsilon$.

Thus, in view of the assertion of Lemma 1, for $x\in\mathcal{K}$, $t\in(0,1)$, and $k\le q_0$, we get

$$\int\limits_{\mathbb{R}^n}|G(t,\zeta)|\left|\partial_{x-\zeta}^k(\eta g)(x-\zeta)-\partial_x^k(\eta g)(x)\right|d\zeta$$

$$\le \varepsilon\int\limits_{||\zeta||<\delta_0}|G(t,\zeta)|\,d\zeta + N_2\int\limits_{||\zeta||\ge\delta_0}|G(t,\zeta)|\,d\zeta$$

$$\le ct^{-\frac{n}{\gamma-\alpha}}\left(\varepsilon\int\limits_{||\zeta||<\delta_0}e^{-\delta||\zeta||/t^\gamma}\,d\zeta + N_2\int\limits_{||\zeta||\ge\delta_0}e^{-\delta||\zeta||/t^\gamma}\,d\zeta\right)$$

$$\le ct^{n(\gamma-\frac{1}{\gamma-\alpha})}\varepsilon\int\limits_{\mathbb{R}^n}e^{-\delta||\zeta||}d\zeta + cN_2t^{-\frac{n}{\gamma-\alpha}}\left(\frac{2t^\gamma}{\delta\delta_0}\right)^{n+1}\sup_{\tau>0}\{\tau^{n+1}e^{-\tau}\}\int\limits_{||\zeta||\ge\delta_0}e^{-\frac{\delta}{2}||\zeta||/t^\gamma}\,d\zeta$$

$$\le (c_0\varepsilon + c_1\delta_0^{-1-n}t^\gamma)\int\limits_{\mathbb{R}^n}e^{-\frac{\delta}{2}||\zeta||}d\zeta \equiv c_0'\varepsilon + c_1'\delta_0^{-1-n}t^\gamma.$$

Note that the quantity t^γ is infinitesimal at the point $t=0$. Hence, for any $\delta_0\in(0,1)$, one can find $t_0\in(0,1)$ such that, for all $t\in(0,t_0)$, the inequality $\delta_0^{1+n}\ge t^{\gamma/2}$ is true. Thus,

$$\forall\varepsilon\in(0,1)\quad \exists t_0\in(0,\varepsilon),\quad \forall t\in(0,t_0)\quad \forall k\in\mathbb{Z}_+^n,\quad k\le q_0:$$

$$\sup_{x\in\mathcal{K}}\left|I_t^k(x)-\partial_x^k(\eta g)(x)\right| \le c_0'\varepsilon + c_1'\varepsilon^{\gamma/2} + N_1O(\varepsilon).$$

Therefore, in view of the equality $\eta g = g$ on $\mathcal{K}$, we arrive at the assertion of the theorem. The theorem is proved.

3. ***Example 1.*** Consider the Cauchy problem for Eq. (1) with $n=1$, $\gamma = 3/2$, and an initial generalized function f from the corresponding space $(W_1^{M_\gamma})'$. The action of this generalized function upon the test functions $\varphi(\cdot) \in W_1^{M_\gamma}$ is specified with the help of the following ordinary function:

$$f(x) = \begin{cases} |x|^{-1/2}, & 0<|x|<1, \\ e^{-x^2}, & |x| \ge 1, \end{cases}$$

according to the rule:

$$\langle f, \varphi(\cdot)\rangle = \int\limits_{|\xi|<1} \frac{\varphi(\xi)}{\sqrt{|\xi|}} d\xi + \int\limits_{|\xi|\ge 1} e^{-\xi^2} \varphi(\xi)\, d\xi.$$

It is clear that this equality enables us to continuously extend the functional f onto the space $S_\beta^{1/(\gamma-\alpha)}$, $\alpha \in (0,\gamma)$. Moreover, f is a convolver in $S_\beta^{1/(\gamma-\alpha)}$. Indeed, for any $\varphi(\cdot) \in S_\beta^{1/(\gamma-\alpha)}$, $\alpha \in (0,\gamma)$, the function

$$\psi(x) := \langle f, \varphi(x-\cdot)\rangle = \int\limits_{|\xi|<1} \frac{\varphi(x-\xi)}{\sqrt{|\xi|}} d\xi + \int\limits_{|\xi|\ge 1} e^{-\xi^2} \varphi(x-\xi)\, d\xi$$

is infinitely differentiable on $\mathbb{R}$ and its derivatives satisfy the estimates

$$\begin{aligned} \left|\psi^{(k)}(x)\right| &\le \int\limits_{|\xi|<1} \frac{\left|\varphi^{(k)}(x-\xi)\right|}{\sqrt{|\xi|}} d\xi + \int\limits_{|\xi|\ge 1} e^{-\xi^2} \left|\varphi^{(k)}(x-\xi)\right| d\xi \\ &\le c A^k k^{\beta k} \left(\int\limits_{|\xi|<1} \frac{e^{-\delta|x-\xi|^{\gamma-\alpha}}}{\sqrt{|\xi|}} d\xi + \int\limits_{|\xi|\ge 1} e^{-\xi^2 - \delta|x-\xi|^{\gamma-\alpha}} d\xi \right) \\ &\le c_0 A^k k^{\beta k} e^{-\delta_0 |x|^{\gamma-\alpha}} \left(\int\limits_{|\xi|<1} \frac{d\xi}{\sqrt{|\xi|}} + \int\limits_{|\xi|\ge 1} e^{-(\xi^2 - \delta_1|\xi|^{\gamma-\alpha})} d\xi \right) \\ &\le c_0' A^k k^{\beta k} e^{-\delta_0 |x|^{\gamma-\alpha}} \end{aligned}$$

for all $k \in \mathbb{Z}_+$, $x \in \mathbb{R}$, and $\alpha \in (0,\gamma)$.

In a similar way, we conclude that f is also a convolver in the space $W_1^{M_\gamma}$.

Hence, the corresponding function

$$u(t,x) = \int_{|\xi|<1} \frac{G(t,x-\xi)}{\sqrt{|\xi|}} d\xi + \int_{|\xi|\geq 1} e^{-\xi^2} G(t,x-\xi)\, d\xi, \qquad (t,x) \in \Pi,$$

is an ordinary solution of Eq. (1) satisfying the corresponding initial condition (4) in a sense of weak convergence in the space $(W_1^{M_\gamma})'$. However, the functional f in the space $(S_1^{5/2})'$ coincides with the smooth functions

$$g_0(\cdot) = |\cdot|^{-1/2} \qquad \text{and} \qquad g_1(\cdot) = e^{-(\cdot)^2}$$

on each set $Q_0 = (-1,0) \cup (0,1)$ and $Q_1 = \mathbb{R}^n \backslash \bar{Q}_0$, respectively. Therefore, according to the assertion of Theorem 2, the limit relations

$$\partial_x^k u(t,x) \underset{t\to+0}{\overset{x\in\mathcal{K}\subset Q_0}{\rightrightarrows}} g_0^{(k)}(x), \qquad \partial_x^k u(t,x) \underset{t\to+0}{\overset{x\in\mathcal{K}\subset Q_1}{\rightrightarrows}} g_1^{(k)}(x)$$

hold for all $k \in \mathbb{Z}_+$ and every compact set $\mathcal{K}$ from the corresponding set Q_j.

CONCLUSIONS

By analyzing the proofs of Theorems 1 and 2, we can make the following conclusions:

The localization principle for the solution of the Cauchy problem for Eq. (1) with generalized limit value from the space $(W_\beta^{M_\gamma})'$, $\beta \geq 1$, can be successfully established if the corresponding space $W_\beta^{M_\gamma}$ of test functions admits an extension by smooth finite functions such that the fundamental solution of the analyzed problem is strongly bounded for $0 < t \ll 1$.

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