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GENERALIZED SPACES OF S AND S' TYPES

In paper the topological structure of generalized spaces of S type and the basic operations in such spaces was investigated. The question of quasi-analyticity (non-quasi-analyticity) of generalized spaces of S type was studied. Some classes of pseudodifferential operators, properties of Fourier transformation of generalized functions from spaces of type S', convolutions, convoluters and multipliers was investigated.

Key words and phrases: the space of generalized function, Fourier transformation, pseudodifferential operator, well-posed solvability, the Cauchy problem, multiplier, convolution, convolutor.

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Introduction

When we work on problem of classes of unity and classes of correctness of the Cauchy problem for equations with partial derivatives with constant or time-dependent coefficients, we often use the spaces of S type, which where introduced by I.M. Gelfand and G.Ye. Shilov in [1]. Spaces of S type (spaces $S_\alpha^\beta \equiv S_{k^{k\alpha}}^{n^{n\beta}}$) are constructed by two sequences $\{k^{k\alpha}\}$, $\{n^{n\beta}\}$, $\{k,n\} \subset \mathbb{Z}_+$ $(0^0 := 1)$, where $\alpha, \beta > 0$ are fixed parameters; elements of such spaces are infinitely differentiable on $\mathbb{R}$ functions φ, which satisfy the condition

$$|x^k \varphi^{(n)}(x)| \leq cA^k B^n k^{k\alpha} n^{n\beta},\ x \in \mathbb{R},\ \{k,n\} \subset \mathbb{Z}_+,$$

with some constants $c, A, B > 0$, dependent on the function φ. Functions from such spaces on the real axis together with all their derivatives at $|x| \to +\infty$ fall faster than $\exp\{-a|x|^{1/\alpha}\}$, $a > 0$, $x \in \mathbb{R}$. Spaces of S and S' types, topologically conjugate with spaces of S type, are natural sets of initial data of the Cauchy problem for the large classes of equations with partial derivatives of finite and infinite orders, in which the solutions are integer functions in terms of spatial variables (see [2, 3, 4, 5, 6]). For example, for the thermal conductivity equation $\partial u/\partial t = \partial^2 u/\partial x^2$ the fundamental solution is the function

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$G(t,x) = (2\sqrt{\pi t})^{-1}\exp\{-x^2/(4t)\}$ for each $t > 0$, as function x, is an element of the $S_{1/2}^{1/2}$ space [5, p. 46], which belongs to spaces of S type.

It is of scientific interest to study $S_{a_k}^{b_n}$ spaces, which are generalizations of S type spaces and are constructed in certain sequences of $\{a_k\}$ and $\{b_n\}$ of positive numbers (study of topological structure, properties of functions, main operators in the specified spaces). This paper provides answers to these questions. The question of quasi-analyticity (non-quasi-analyticity) of generalized spaces of S type is also studied. Some classes of pseudodifferential operators in such spaces, properties of Fourier transform of generalized functions from spaces of S' type, convolutions, convoluters and multipliers are investigated. The obtained results were used in the study of the Cauchy problem for the evolution equation with the fractional differentiation operator $A = \sqrt{I - \partial^2/\partial x^2}$ and the initial function, which is an element of the space of generalized functions such as ultradistributions.

1 Preliminaries. Topological structure of generalized spaces of S type

Consider the sequence $\{m_n, n \in \mathbb{Z}_+\}$ of positive numbers, which has the properties:
1) $\forall n \in \mathbb{Z}_+ : m_n \le m_{n+1},\ m_0 = 1$;
2) $\exists M > 0\ \exists h > 0\ \forall n \in \mathbb{Z}_+ : m_{n+1} \le M h^n m_n$;
3) $\exists c_1 \ge 1\ \exists L \ge 1 : m_k \cdot m_{n-k} \le c_1 L^n m_n,\ k \in \{0, 1, ..., n\}$.
Examples of such sequences are the Gevrey sequences of the form $m_n = n^{n\beta}$, $m_n = (n!)^\beta$, $n \in \mathbb{Z}_+$, where $\beta > 0$ is fixed parameter.

Let
$$\gamma(x) := \inf_{n\in\mathbb{Z}_+} \frac{m_n}{|x|^n},\ x \ne 0.$$
It is obvious that γ is a non-negative, even function on $\mathbb{R}\setminus\{0\}$. If $x \in [-1,1]\setminus\{0\}$, then, taking into account property 1) of the sequence $\{m_n, n \in \mathbb{Z}_+\}$ we have, $\inf\limits_{n\in\mathbb{Z}_+} \frac{m_n}{|x|^n} = 1$, ie $\gamma(x) = 1$ for $x \in [-1,1]\setminus\{0\}$.

If $1 \le x_1 < x_2$, then $\gamma(x_2) < \gamma(x_1) \le \gamma(1) = 1$, ie γ monotonically falls on the interval $[1, +\infty)$. Hence, taking into account the parity property of the function γ on $\mathbb{R}\setminus\{0\}$ we get that γ grows monotonically on the interval $(-\infty, -1]$, $0 < \gamma(x) \le 1$, $\forall x \in \mathbb{R}\setminus\{0\}$.

For example, if $m_n = n^{n\alpha}$, $n \in \mathbb{Z}_+$, $\alpha > 0$, then in [1, p. 205] established the following assessment:
$$\gamma_\alpha(\xi) := \inf_{n\in\mathbb{Z}_+} \frac{n^{n\alpha}}{|\xi|^n} \le e^{\alpha e/2} \cdot e^{-\frac{\alpha}{e}\xi^{1/\alpha}},\ \xi \ge 1.$$
If $0 < \xi < 1$, then
$$\inf_{n\in\mathbb{Z}_+} \frac{n^{n\alpha}}{\xi^n} = 1 \le e^{\frac{\alpha}{e}} \cdot e^{-\frac{\alpha}{e}\xi^{1/\alpha}}.$$
So,
$$\forall \xi : 0 < \xi < +\infty :\ \gamma_\alpha(\xi) \le c e^{-\frac{\alpha}{e}\xi^{1/\alpha}},\ c = e^{\alpha e/2}.$$
In addition, on $\mathbb{R}\setminus\{0\}$ the function γ_α satisfies the inequalities [1, pp. 204]:
$$e^{-\frac{\alpha}{e}|\xi|^{1/\alpha}} \le \inf_{n\in\mathbb{Z}_+} \frac{n^{n\alpha}}{|\xi|^n} \le c e^{-\frac{\alpha}{e}|\xi|^{1/\alpha}},\ c = e^{\alpha e/2},\ \xi \in \mathbb{R}\setminus\{0\}. \tag{1}$$

Lemma 1. *The inequality*

$$\ln\gamma(x_1)+\ln\gamma(x_2)\geq\ln\gamma(x_1+x_2),\ \forall\{x_1,x_2\}\subset(0,+\infty), \tag{2}$$

is correct.

Proof. First of all, note that $\{\gamma(x_1),\ \gamma(x_2),\ \gamma(x_1+x_2)\}\subset(0,1]$ for arbitrary fixed $\{x_1,x_2\}\subset(0,+\infty)$. Since $\gamma(x)=1$ for $x\in(0,1]$, so it is enough to prove the inequality (2) on the interval $(1,+\infty)$. Indeed, if $\{x_1,x_2\}\subset(0,1]$ and $(x_1+x_2)\in(0,1]$, then the inequality (2) becomes equality. If $\{x_1,x_2\}\subset(0,1]$ and $x_1+x_2>1$, then the inequality (2) also holds, because $0<\gamma(x_1+x_2)<1$, $\ln\gamma(x_1+x_2)<0$, and $\gamma(x_1)=\gamma(x_2)=1$ and $\ln\gamma(x_1)=\ln\gamma(x_2)=0$. If $x_1\in(0,1]$, and $x_2>1$, then $x_1+x_2>1$, $\ln\gamma(x_1)=0$, $\ln\gamma(x_1)+\ln\gamma(x_2)=\ln\gamma(x_2)\geq\ln\gamma(x_1+x_2)$, since $\gamma(x_1+x_2)\leq\gamma(x_2)$ (here it is taken into account that γ monotonically falls on the interval $(1,+\infty)$). Similarly consider the case when $x_2\in[0,1]$, $x_1>1$.

So let $\{x_1,x_2\}\subset(1,+\infty)$. The inequality (2) is equivalent to an inequality

$$\gamma(x_1)\cdot\gamma(x_2)\geq\gamma(x_1+x_2),\ \{x_1,x_2\}\subset(1,+\infty). \tag{3}$$

To prove (3) it is enough to establish that

$$\frac{\gamma(x_1)\cdot\gamma(x_2)}{\gamma(x_1+x_2)}\geq 1,\ \{x_1,x_2\}\subset(1,+\infty).$$

Let $1<x_1\leq x_2$. Since γ monotonically decreases to $(1,+\infty)$, then $\gamma(x_1)\geq\gamma(x_2)$. So,

$$\frac{\gamma(x_1)\cdot\gamma(x_2)}{\gamma(x_1+x_2)}\geq\frac{\gamma^2(x_2)}{\gamma(x_1+x_2)}.$$

By definition, $\gamma(x_2)=\inf\limits_{n\in\mathbb{Z}_+}\frac{m_n}{x_2^n}$, $x_2\in(1,+\infty)$. Consider the sequence $\{\varepsilon_k=\beta_k\gamma(x_2),k\in\mathbb{N}\}$, where the sequence $\{\beta_k,\ k\in\mathbb{N}\}$ of positive numbers monotonically tends to zero for $k\to+\infty$. Then for $\varepsilon_k>0$ there is a number $n_k=n_k(\varepsilon_k)$ such that

$$\frac{m_{n_k}}{x_2^{n_k}}<\gamma(x_2)+\varepsilon_k=(1+\beta_k)\gamma(x_2),$$

that is

$$\gamma(x_2)>\frac{1}{1+\beta_k}\frac{m_{n_k}}{x_2^{n_k}},\ k\in\mathbb{N}.$$

In accordance,

$$\gamma(x_1+x_2)\leq\frac{m_{n_k}}{(x_1+x_2)^{n_k}},\ k\in\mathbb{N}.$$

Given these inequalities, we conclude that for the numbers $\{n_k,\ k\in\mathbb{N}\}$ the inequalities hold

$$\frac{\gamma(x_1)\gamma(x_2)}{\gamma(x_1+x_2)}\geq\frac{\gamma^2(x_2)}{\gamma(x_1+x_2)}\geq\frac{m^2_{n_k}}{(1+\beta_k)^2}\frac{(x_1+x_2)^{n_k}}{x_2^{2n_k}m_{n_k}}\geq\frac{m_{n_k}}{(1+\beta_1)^2x_2^{n_k}},\ k\in\mathbb{N}$$

(it is taken into account that $x_1+x_2\geq x_2$, $\beta_k<\beta_1$, $\forall k\geq 2$). In addition,

$$\gamma(\alpha)\leq\frac{m_n}{\alpha^n},\ n\in\mathbb{Z}_+,$$

for any $\alpha > 1$, or

$$\forall \alpha > 1\ \forall k \in \mathbb{N}:\ m_{n_k} \geq \alpha^{n_k}\gamma(\alpha).$$

Put $\alpha = x_2\delta$, where $\delta > 1$ is a fixed number and choose the number n_k so that the inequality $\delta^{n_k}\gamma(x_2\delta) \geq (1+\beta_1)^2$ is true. Directly find that

$$n_k \geq \left[\ln\left(\frac{(1+\beta_1)^2}{\gamma(x_2\delta)}\right)(\ln\delta)^{-1} + 1\right].$$

For such a number, the inequality holds

$$\frac{\gamma(x_1)\gamma(x_2)}{\gamma(x_1+x_2)} \geq \frac{x_2^{n_k}\delta^{n_k}\gamma(x_2\delta)}{(1+\beta_1)^2 x_2^{n_k}} = \frac{\delta^{n_k}\gamma(x_2\delta)}{(1+\beta_1)^2} \geq 1,$$

which had to be proved. □

Let $\{a_k,\ k \in \mathbb{Z}_+\}$ and $\{b_n,\ n \in \mathbb{Z}_+\}$ are the sequences that have properties 1) - 3). The symbol $S_{a_k}^{b_n}$ denote the set of functions $\varphi \in C^\infty(\mathbb{R})$ that satisfy the condition

$$\exists\, c, A, B > 0\ \forall \{k,n\} \subset \mathbb{Z}_+\ \forall x \in \mathbb{R} : |x^k\varphi^{(n)}(x)| \leq cA^kB^na_kb_n \tag{4}$$

(constants $c, A, B > 0$ depend on the function φ).

$S_{a_k}^{b_n}$ coincides with the union of spaces $S_{a_k,A}^{b_n,B}$ for all $A, B > 0$, where the symbol $S_{a_k,A}^{b_n,B}$ denotes the set of functions $\varphi \in S_{a_k}^{b_n}$, which for arbitrary $\delta, \rho > 0$ satisfy the inequalities

$$|x^k\varphi^{(n)}(x)| \leq c_{\delta\rho}(A+\delta)^k(B+\rho)^na_kb_n,\ \{k,n\} \subset \mathbb{Z}_+,\ x \in \mathbb{R},$$

with the same constants $A, B > 0$. $S_{a_k,A}^{b_n,B}$ is transformed into a complete countable-normalized space, if the system of norms in this space is given by formulas

$$\|\varphi\|_{\delta,\rho} = \sup_{x,k,n}\frac{|x^k\varphi^{(n)}(x)|}{(A+\delta)^k(B+\rho)^na_kb_n},\ \varphi \in S_{a_k,A}^{b_n,B},\ \{\delta,\rho\} \subset \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}.$$

The sequence $\{\varphi_\nu,\ \nu \in \mathbb{N}\} \subset S_{a_k}^{b_n}$ goes to zero in the space $S_{a_k}^{b_n}$ for $\nu \to +\infty$, if $\{\varphi_\nu,\ \nu \in \mathbb{N}\} \subset S_{a_k,A}^{b_n,B}$ for some $A, B > 0$ and goes to zero in this space, so $\|\varphi_\nu\|_{\delta\rho} \to 0$ for $\nu \to \infty$ for all $\{\delta,\rho\} \subset \{1, \frac{1}{2}, \ldots\}$. This definition is equivalent to this: the sequence $\{\varphi_\nu,\ \nu \in \mathbb{N}\} \subset S_{a_k}^{b_n}$ goes to zero in this space, if the functions φ_ν and their derivatives of arbitrary order go to zero uniformly on each $[a,b] \subset \mathbb{R}$ and at the same time inequalities come true

$$|x^k\varphi_\nu^{(n)}(x)| \leq cA^kB^na_kb_n,\ \{k,n\} \subset \mathbb{Z}_+,\ x \in \mathbb{R},$$

where constants $c, A, B > 0$ do not depend on ν (the proof of this statement is similar to the proof of a similar statement in the case of the S_α^β spaces (see [1, p. 219])).

The set $F \subset S_{a_k}^{b_n}$ is called *bounded*, if F is contained in the $S_{a_k,A}^{b_n,B}$ space with the some values $A, B > 0$ and is bounded in this space, so that for all functions $\varphi \in F$ the evaluation (4) is true with the same constants $c, A, B > 0$.

Lemma 2. *The function $\varphi \in C^\infty(\mathbb{R})$ is an element of the space $S_{a_k}^{b_n}$ if and only if it satisfies the condition*

$$\exists a, B, c > 0\ \forall n \in \mathbb{Z}_+\ \forall x \in \mathbb{R} : |\varphi^{(n)}(x)| \le cB^n b_n \widetilde{\gamma}(ax), \tag{5}$$

where

$$\widetilde{\gamma}(x) = \begin{cases} 1, & |x| \le 1, \\ \inf\limits_{k \in \mathbb{Z}_+} \frac{a_k}{|x|^k}, & |x| > 1. \end{cases}$$

Proof. Let $\varphi \in S_{a_k}^{b_n}$, ie the condition (4) holds. Then, dividing both parts of the inequality (4) by $|x|^k$, $x \neq 0$, and in the right part going to the lower limit of k, we obtain

$$|\varphi^{(n)}(x)| \le cB^n b_n \inf_{k \in \mathbb{Z}_+} \frac{A^k a_k}{|x|^k} = cB^n b_n \inf_{k \in \mathbb{Z}_+} \frac{a_k}{|A^{-1}x|^k} = cB^n b_n \gamma(ax),\ x \neq 0,$$

where $a = A^{-1} > 0$. Since $|\varphi^{(n)}(0)| \le cB^n b_n$, $n \in \mathbb{N}$ (see (4)) and $\gamma(x) = 1$ for $x \in [-1, 1] \setminus \{0\}$, then the function γ in the last inequality can be replaced by $\widetilde{\gamma}$.

Conversely, let the function $\varphi \in C^\infty(\mathbb{R})$ satisfy the condition (5). Then

$$|\varphi^{(n)}(x)| \le cB^n b_n \inf_{k \in \mathbb{Z}_+} \frac{a_k}{|ax|^k},\ n \in \mathbb{Z}_+,\ x \neq 0,$$

hence it follows that the inequality

$$\forall x \in \mathbb{R} \setminus \{0\}:\ |ax|^k |\varphi^{(n)}(x)| \le cB^n b_n a_k,\ \{k, n\} \subset \mathbb{Z}_+$$

holds. Given the estimate $|\varphi^{(n)}(0)| \le cB^n b_n$, $n \in \mathbb{Z}_+$, we have

$$|x^k \varphi^{(n)}(x)| \le cA^k B^n a_k b_n,\ A = a^{-1},\ \forall \{k, n\} \subset \mathbb{Z}_+,\ x \in \mathbb{R},$$

which was to be proved. □

If $a_k = k^{k\alpha}$, $b_n = n^{n\beta}$, $\{k, n\} \subset \mathbb{Z}_+$, where α, $\beta > 0$ are fixed parameters, then in this case the space $S_{k^{k\alpha}}^{n^{n\beta}}$ is denoted by the symbol S_α^β. S_α^β spaces are called S type spaces; there is a lot of detail in the monograph [1], which can be characterized as follows [1, p. 210].

The spaces S_α^β are nontrivial if $\alpha + \beta \ge 1$, $\alpha, \beta > 0$ and form dense sets in $L_2(\mathbb{R})$.

S_α^β, $\alpha > 0, \beta > 0$, $\alpha + \beta \ge 1$, consists of those and only those infinitely differentiable on $\mathbb{R}$ functions that satisfy the inequalities

$$|\varphi^{(n)}(x)| \le cB^n n^{n\beta} \exp\{-a|x|^{1/\alpha}\},\ n \in \mathbb{Z}_+,\ x \in \mathbb{R},$$

with some constants c, A, $B > 0$, dependent only on the function φ.

If $0 < \beta < 1$ and $\alpha \ge 1 - \beta$, then S_α^β consists of those and only those functions $\varphi \in C^\infty(\mathbb{R})$, which analytically extend into the whole complex plane and satisfy the condition

$$\exists c = c(\varphi) > 0\ \exists a = a(\varphi) > 0\ \exists b = b(\varphi) > 0:$$

$$|\varphi(x + iy)| \le c \exp\{-a|x|^{1/\alpha} + b|y|^{1/(1-\beta)}\},\ \forall \{x, y\} \subset \mathbb{R}.$$

The space S^1_α ($\alpha > 0$ is arbitrary) consists of functions $\varphi \in C^\infty(\mathbb{R})$, which analytically extend to the function $\varphi(x+iy)$ in some band $|y| < \delta$ (dependent on φ) of the complex plane, while

$$|\varphi(x+iy)| \leq c\exp\{-a|x|^{1/\alpha}\},\ c, a > 0,\ \{x, y\} \subset \mathbb{R},\ |y| < \delta.$$

The spaces $S^{b_n}_{a_k}$, constructed by the sequences $\{a_k\}$, $\{b_n\}$, which satisfy conditions 1) - 3), will be called generalized spaces of S type.

In the spaces $S^{b_n}_{a_k}$ are defined, are linear and continuous operators of argument shift, multiplication by an independent variable and differentiation.

We prove, for example, that in the space $S^{b_n}_{a_k}$ is defined and is a continuous operator of the shift of the argument T_{-h}: $\varphi(x) \to \varphi(x-h)$, $\forall \varphi \in S^{b_n}_{a_k}$, which reflects this space in itself.

Let φ run over a bounded set $F \subset S^{b_n}_{a_k}$. This means that for each function $\varphi \in F$ inequalities

$$|x^k\varphi^{(n)}(x)| \leq cA^kB^na_kb_n,\ x \in \mathbb{R},\ \{k, n\} \subset \mathbb{Z}_+,$$

hold with the same constants $c,\ A,\ B > 0$. Then

$$\sup_{x\in\mathbb{R}} |x^k\varphi^{(n)}(x-h)| = \sup_{x\in\mathbb{R}} |(x+h)^k\varphi^{(n)}x| = \sup_{x\in\mathbb{R}} |\sum_{j=0}^{k} C^j_k x^j h^{k-j}\varphi^{(n)}(x)| \leq$$

$$\leq \sum_{j=0}^{k} C^j_k|h|^{k-j} \sup_{x\in\mathbb{R}} |x^j\varphi^{(n)}(x)| \leq c\sum_{j=0}^{k} C^j_k|h|^{k-j}A^jB^na_jb_n \leq cB^nb_na_k\sum_{j=0}^{k} C^j_kA^j|h|^{k-j} =$$

$$= c(A+|h|)^kB^na_kb_n = cA_1^kB^na_kb_n.$$

where $A_1 = A + |h|$. It follows that the function $\varphi_1(x) = \varphi(x-h)$ is an element of the space $S^{b_n,B}_{a_k,A+|h|}$, ie $\varphi_1 \in S^{b_n}_{a_k} = \bigcup\limits_{A,B>0} S^{b_n,B}_{a_k,A}$. Therefore, the image of the bounded set F at the specified mapping is a bounded set in the space $S^{b_n}_{a_k}$. This means that the argument shift operator is a linear bounded operator in the space $S^{b_n}_{a_k}$, and hence a linear continuous operator in this space, because in the space $S^{b_n}_{a_k}$ is executed the first axiom of countability. Then, as follows from the general theory of linear continuous operators in countable-normalized spaces (see [1, pp. 81-82]), in spaces with the first axiom of countability the class of linear bounded operators coincides with the class of linear continuous operators.

Note also that the spaces $S^{b_n}_{a_k}$ are perfect (that is, spaces whose bounded sets are compact). It follows from this and from the general theory of perfect spaces (see [1, p. 171]) that the operation of shifting an argument is differential (even infinitely differentiable) in the sense that the boundary relations of the form $(\varphi(x+h) - \varphi(x))h^{-1} \to \varphi'(x)$, $h \to 0$, are valid for each function $\varphi \in S^{b_n}_{a_k}$ in the sense of topology convergence space $S^{b_n}_{a_k}$.

We prove that in the space $S^{b_n}_{a_k}$ is defined, there is a linear and continuous multiplication operator for an independent variable that reflects this space in itself.

Let φ run over a bounded set $F \subset S^{b_n}_{a_k}$, that is, every function $\varphi \in F$ satisfies the inequalities

$$|x^k\varphi^{(n)}(x)| \leq cA^kB^na_kb_n,\ \{k, n\} \subset \mathbb{Z}_+,\ x \in \mathbb{R},$$

with some constants $c, A, B > 0$. Put $\psi(x) := x\varphi(x)$. Then

$$|x^k\psi^{(n)}(x)| = |x^k(x\varphi(x))^{(n)}| \le |x^{k+1}\varphi^{(n)}(x)| + n|x^k\varphi^{(n-1)}(x)| \le$$

$$\le cA^{k+1}B^n a_{k+1}b_n + ncA^kB^{n-1}a_kb_{n-1}.$$

Using property 2) of the sequence $\{a_k\}$ and property 1) of the sequence $\{b_n\}$, we arrive at the inequalities

$$|x^k\psi^{(n)}(x)| \le cAA^kB^nMh^ka_kb_n + c2^nA^kB^nB^{-1}a_kb_n = \widetilde{c}A_1^kB_1^na_kb_n,$$

where $\widetilde{c} = cAM + cB^{-1}$, $A_1 = \max\{Ah, A\}$, $B_1 = 2\max\{1, 2B\}$. Thus, the image of the bounded set F when multiplied by the independent variable x is again a bounded set in the space $S_{a_k}^{b_n}$, which was to be proved.

Note also that $\varphi\psi \in S_{a_k}^{b_n}$ for arbitrary $\{\varphi, \psi\} \subset S_{a_k}^{b_n}$.

The function $g \in C^\infty(\mathbb{R})$ is called the *multiplier* in the space $S_{a_k}^{b_n}$, if $g\varphi \in C^\infty(\mathbb{R})$ for an arbitrary function $\varphi \in S_{a_k}^{b_n}$ and the mapping $\varphi \to g\varphi$ is linear and continuous.

Lemma 3. *The multiplier in the space $S_{a_k}^{b_n}$ is the function $f \in C^\infty(\mathbb{R})$, which satisfies the condition*

$$\exists B_0\ \forall \varepsilon > 0\ \exists c_\varepsilon > 0\ \forall n \in \mathbb{Z}_+\ \forall x \in \mathbb{R}:\ |f^{(n)}(x)| \le c_\varepsilon B_0^n b_n(\widetilde{\gamma}(\varepsilon x))^{-1}. \tag{6}$$

Proof. Let $\varphi \in S_{a_k}^{b_n}$. Then, according to Lemma 2, the inequalities are correct

$$|f^{(n)}(x)| \le cB^nb_n\widetilde{\gamma}(ax),\ x \in \mathbb{R},\ n \in \mathbb{Z}_+,$$

with some constants $c,\ a,\ B > 0$. Take $\varepsilon \in (0, a)$ and use the estimates (6). Then

$$|(f(x)\varphi(x))^{(n)}| \le \sum_{j=0}^{n} C_n^j|f^{(i)}(x)| \cdot |\varphi^{(n-j)}(x)| \le cc_\varepsilon \sum_{j=0}^{n} C_n^jB_0^jB^{(n-j)}b_jb_{n-j}\frac{\widetilde{\gamma}(ax)}{\widetilde{\gamma}(\varepsilon x)}.$$

Since $b_jb_{n-j} \le \omega L^nb_n$ (see property 3) of the sequence $\{b_n\}$), then

$$|(f(x)\varphi(x))^{(n)}| \le \widetilde{c}B_1^nb_n\frac{\widetilde{\gamma}(ax)}{\widetilde{\gamma}(\varepsilon x)} = \widetilde{c}B_1^nb_ne^{\ln\widetilde{\gamma}(ax) - \ln\widetilde{\gamma}(\varepsilon x)},$$

where $\widetilde{c} = cc_\varepsilon\omega$, $B_1 = 2\max\{B_0, B\}L$. From (2) follows the inequality

$$\ln\widetilde{\gamma}(ax) - \ln\widetilde{\gamma}(\varepsilon x) \le \ln\widetilde{\gamma}((a-\varepsilon)x),\ 0 < \varepsilon < a.$$

Then

$$|(f(x)\varphi(x))^{(n)}| \le \widetilde{c}B_1^nb_ne^{\ln\widetilde{\gamma}((a-\varepsilon)x)} = \widetilde{c}B_1^nb_n\widetilde{\gamma}(a_1x),\ a_1 = a - \varepsilon.$$

Therefore, $f\varphi \in S_{a_k}^{b_n}$. It also follows from the above considerations that if φ flows through a bounded set $F \subset S_{a_k}^{b_n}$, then each function $f\varphi$, $\varphi \in F$, belongs to a limited set $F_1 \subset S_{a_k}^{b_n}$, ie the operator $S_{a_k}^{b_n} \ni \varphi \to f\varphi \in S_{a_k}^{b_n}$ is continuous. The Lemma is proved. □

If the sequences $\{a_k\}$, $\{b_n\}$ satisfy the conditions

$$\frac{a_k}{a_{k-1}} \geq c_a k^{1-\mu}, \ \frac{b_k}{b_{k-1}} \geq c_b k^{1-\lambda}, \ \mu, \lambda \geq 0, \ \mu + \lambda \leq 1, \ \{k, n\} \subset \mathbb{N}, \ \frac{a_{k+2}}{a_k} \leq c_0 A^k, \tag{7}$$

then, as noted in [1, p. 290], the formula $F[S_{a_k}^{b_n}] = S_{b_k}^{a_n}$ is correct, where $F[S_{a_k}^{b_n}] := \left\{\psi : \psi(\sigma) = \int\limits_{\mathbb{R}} \varphi(\sigma) e^{i\sigma x} dx, \ \forall \varphi \in S_{a_k}^{b_n}\right\}$, in particular,

$$F[S_{k^{k\alpha}}^{n^{n\beta}}] \equiv F[S_\alpha^\beta] = S_\beta^\alpha \equiv S_{k^{k\beta}}^{n^{n\alpha}}, \ \alpha, \beta > 0.$$

Note that since the sequence $\{a_k, k \in \mathbb{Z}_+\}$ has property 2), the inequality $\frac{a_{k+2}}{a_k} \leq c_0 A_0^k$ holds with constants $c_0 = hM^2, \ A_0 = h^2$.

2 On the quasi-analyticity of generalized spaces of S type

If the sequence $\{b_n, n \in \mathbb{Z}_+\}$, which is used to construct the space $S_{a_k}^{b_n}$, "grows slowly", then such space can consist of infinite differentiable on $\mathbb{R}$ functions, which allow analytical continuation in the whole complex plane and satisfy a certain condition.

Theorem 1. *Suppose there exists $L \in [0, +\infty)$ such that*

$$\lim_{n\to\infty} \frac{\sqrt[n]{b_n}}{n} = L.$$

The following statements are correct:

1. If $L \in (0, +\infty)$, then each function $\varphi(x)$ from the space $S_{a_k}^{b_n}$ allows an analytical extension in some band $|Imz| = |Im(x+iy)| = |y| < c$, $c = c(\varphi) > 0$, of the complex plane.

2. If $L = 0$, then each function $\varphi(x)$ from the space $S_{a_k}^{b_n}$ analytically extends into the whole complex plane to the whole function $\varphi(x+iy)$, which satisfies the inequality

$$|\varphi(x+iy)| \leq c\widetilde{\gamma}(ax)\widetilde{\rho}(by), \ \forall \{x, y\} \subset \mathbb{R},$$

where $c, \ a, \ b > 0$ are some constants (dependent on φ),

$$\widetilde{\gamma}(x) = \begin{cases} 1, & |x| \leq 1, \\ \inf\limits_{k\in\mathbb{Z}_+} \frac{a_k}{|x|^k}, & |x| > 1, \end{cases} \qquad \widetilde{\rho}(y) = \begin{cases} 1, & |y| < 1, \\ \sup\limits_{n\in\mathbb{Z}_+} \frac{|y|^n}{\widehat{b}_n}, & |y| > 1, \ \widehat{b}_n = \frac{n!}{b_n}. \end{cases}$$

3. In the case where there is $\omega \in (1, +\infty)$ such that $\lim\limits_{n\to\infty} \frac{\sqrt[n]{b_n}}{n^\omega} = +\infty$, among the elements of the space $S_{a_k}^{b_n}$ there are finite infinitely differentiable functions.

Proof. Let fixed an arbitrary function φ from the space $S_{a_k}^{b_n}$ and estimate its residual term in the form of Taylor

$$\left|\frac{h^n}{n!}\varphi^{(n)}(x+\theta h)\right| \leq c\frac{|h|^n}{n!} B^n b_n, \ n \in \mathbb{Z}_+, \ \{x, \, h\} \subset \mathbb{R}, \ 0 < \theta < 1.$$

If the condition $\lim\limits_{n\to\infty}\frac{\sqrt[n]{b_n}}{n}=L$, $0<L<\infty$, is true then

$$\forall\varepsilon>0\ \exists n_0=n_0(\varepsilon)\in\mathbb{N}\ \forall n\geq n_0:\ b_n<(\varepsilon+L)^n n^n.$$

From Stirling's formula $n!=\sqrt{2\pi n}n^ne^{-n}e^{\theta_n/(1-n)},\ 0<\theta_n<1$, the inequality $n!\geq n^ne^{-n}$ follows. From here we get the going to zero of the residual term at $n\to\infty$ for all $h:\ |h|<(BLe)^{-1}$. Thus, in the corresponding neighborhood of the point x the function φ develops in a Taylor series converging to it

$$\varphi(x+h)=\sum_{n=0}^{\infty}\frac{h^n}{n!}\varphi^{(n)}(x).$$

Since this series are also convergent for complex values of h such that $|h|<(BLe)^{-1}$, we conclude that the function φ admits an analytic extension to the band $|h|<(BLe)^{-1}$ of the complex plane.

If $L=0$, then

$$\forall\varepsilon>0\ \exists n_0=n_0(\varepsilon)\in\mathbb{N}\ \forall n\geq n_0:\ b_n<(\varepsilon e)^n n^n.$$

Let fix arbitrary $|h|\neq 0$ and put $\varepsilon=\frac{1}{2}(Be|h|)^{-1}$. Then

$$\left|\frac{h^n}{n!}\varphi^{(n)}(x+\theta h)\right|\leq\frac{c}{2^n}\to 0,\ n\to\infty.$$

Therefore, the Taylor series of the function φ converging for an arbitrary complex h. Putting $h=iy$, $y\in\mathbb{R}\setminus\{0\}$, we get that the function $\varphi(x)$ analytically extends in the whole complex plane to the function

$$\varphi(x+iy)=\sum_{n=0}^{\infty}\frac{(iy)^n}{n!}\varphi^{(n)}(x),\ y\neq 0.$$

From here and from the inequality (5) we get

$$|\varphi(x+iy)|\leq c\sum_{n=0}^{\infty}\frac{|y|^n}{n!}B^nb_n\widetilde{\gamma}(ax)\leq c\sup_{n\in\mathbb{Z}_+}\left(\frac{2^nB^n|y|^n}{n!}b_n\right)\sum_{n=0}^{\infty}\frac{1}{2^n}\widetilde{\gamma}(ax)=$$

$$=c_1\widetilde{\gamma}(ax)\widetilde{\rho}_1(by),\ c_1=2c,\ b=2B,\ y\neq 0,$$

where $\widetilde{\rho}_1(y)=\sup\limits_{n\in\mathbb{Z}_+}\frac{|y|^n}{\widehat{b}_n}$, $\widehat{b}_n=\frac{n!}{b_n}$, $y\neq 0$. If $y=0$, then $|\varphi(x)|\leq c\widetilde{\gamma}(ax)$. Note when $L=0$, then the sequence $\frac{b_n}{n!}$ monotonically tends to zero for $n\to\infty$. Thus, the sequence $\widehat{b}_n=\frac{n!}{b_n}$ monotonically tends to infinity for $n\to\infty$. Since, $\widehat{b}_0=1$ and $\widetilde{\rho}_1(y)=1$ for $y\in[-1,1]\setminus\{0\}$, $\widetilde{\rho}_1(0)=0$, then instead of the function $\widetilde{\rho}_1$ we can consider the function

$$\widetilde{\rho}(y)=\begin{cases}1, & |y|<1,\\ \sup\limits_{n\in\mathbb{Z}_+}\frac{|y|^n}{\widehat{b}_n}, & |y|>1.\end{cases}$$

3. The condition $\lim\limits_{n\to\infty} \frac{\sqrt[n]{b_n}}{n^\omega} = +\infty$ (for some $\omega > 1$) implies the existence of a constant $A > 1$ such that the inequality $b_n \geq A^n n^{n\omega}$ for all $n \in \mathbb{N}$ is true. Next we use the inequality

$$\inf_{n\in\mathbb{Z}_+} \frac{A^n n^{n\alpha}}{|x|^n} \geq \exp\left\{-\frac{\alpha}{e}\left|\frac{x}{A}\right|^{1/\alpha}\right\},\ x \neq 0,$$

where $\alpha, A > 0$ are fixed numbers, which follows from the estimates (1). Given the relation

$$\sup_{n\in\mathbb{Z}_+} \frac{|x|^n}{A^n n^{n\alpha}} = \frac{1}{\inf\limits_{n\in\mathbb{Z}_+} \frac{A^n n^{n\alpha}}{|x|^n}},\ x \neq 0,$$

come to the assessment

$$\sup_{n\in\mathbb{Z}_+} \frac{|x|^n}{A^n n^{n\alpha}} \leq \exp\left\{\frac{\alpha}{e}\left|\frac{x}{A}\right|^{1/\alpha}\right\},\ x \in \mathbb{R}. \tag{8}$$

If we put in (8) $\alpha = \omega > 1$ and taking into account the inequality $b_n \geq A^n n^{n\omega}, n \in \mathbb{N}$, we obtain

$$T(\lambda) := \sup_{n\in\mathbb{Z}_+} \frac{\lambda^n}{b_n} \leq \sup_{n\in\mathbb{Z}_+} \frac{\lambda^n}{A^n n^{n\omega}} \leq \exp\{\widetilde{a}\lambda^{1/\omega}\},\ \widetilde{a} = \omega e/A^{1/\omega},\ \lambda \geq 1.$$

Then

$$\int_1^{+\infty} \frac{\ln T(\lambda)}{\lambda^2} d\lambda \leq \widetilde{a} \int_1^{+\infty} \lambda^{1/\omega - 2} d\lambda = \frac{\omega - 1}{eA^{1/\omega}} < +\infty.$$

Hence and from the Carleman-Ostrovsky theorem [7], which describes the classes of quasi-analytic (non-quasi-analytic) functions, it follows that among the elements of the space $S_{a_k}^{b_n}$ there are finite infinitely differentiable functions.

Theorem proved. □

As an example of the application of Theorem 1, consider the space $S_\alpha^\beta = S_{k^{k\alpha}}^{n^{n\beta}}$, where $\alpha > 0$, $\beta > 0$, $\alpha \geq 1 - \beta$ (condition of non-triviality of space S_α^β). In this case $b_n = n^{n\beta}$, $n \in \mathbb{Z}_+$.

If $\beta \in (0, 1)$, then

$$\widetilde{\rho}_1(y) = \sup_{n\in\mathbb{Z}_+} \frac{|y|^n}{\widehat{b}_n} \leq \sup_{n\in\mathbb{Z}_+} \frac{|ey|^n}{n^{n(1-\beta)}} = \frac{1}{\inf\limits_{n\in\mathbb{Z}_+} \frac{n^{n(1-\beta)}}{|ey|^n}},\ y \neq 0.$$

From the estimates (1) we obtain inequalities

$$\widetilde{\rho}(y) \leq \exp\{b|y|^{1/(1-\beta)}\},\ b > 0,\ \widetilde{\gamma}(x) \leq c\exp\{-a|x|^{1/\alpha}\},\ c > 1.$$

Therefore, it follows from this and from Proposition 2 of Theorem 1 that each function $\varphi \in S_\alpha^\beta$, $\alpha > 0$, $\beta \in (0,1)$, $\alpha \geq 1 - \beta$, analytically extends into the whole complex plane and satisfies the inequality

$$|\varphi(x + iy)| \leq c\exp\{-a|x|^{1/\alpha} + b|y|^{1/(1-\beta)}\},$$

where $c,\ a,\ b > 0$ are some constants depending on the function φ (obtained a known result established in [1, p. 209]).

If $\beta = 1$, then $b_n = n^n$, $\lim\limits_{n\to\infty} \frac{\sqrt[n]{b_n}}{n} = 1$. In this case, each function $\varphi \in S^1_\alpha$, $\alpha > 0$, analytically extends into some band of the complex plane, the width of which depends on the function φ.

If $\beta > 1$, then for an arbitrary fixed $\omega \in (1, \beta)$ we have $\frac{\sqrt[n]{b_n}}{n^\omega} = n^{\beta-\omega}$, $\lim\limits_{n\to\infty} \frac{\sqrt[n]{b_n}}{n^\omega} = +\infty$. Thus, the space S^β_α, $\alpha > 0$, $\beta > 1$, contains finite infinitely differentiable functions on $\mathbb{R}$ (the same result in the case of the space S^β_α for $\beta > 1$, follows directly from the Carleman-Ostrovsky theorem).

3 Pseudodifferential operators in generalized spaces of S type

The symbol $\Theta^{b_n}_{a_k}$ denotes the set of functions $\varphi \in C^\infty(\mathbb{R})$, which are multipliers in the space $S^{b_n}_{a_k}$. From the properties of the Fourier transform (direct and inverse) in generalized spaces of type S it follows that in the space $S^{a_n}_{b_k}$ is defined, is a linear and continuous operator $A := F^{-1}_{\sigma\to x}[\varphi(\sigma)F_{x\to\sigma}]$, which is called a pseudodifferential operator built on the function $\varphi \in \Theta^{b_n}_{a_k}$ (operator symbol A), $A : S^{a_n}_{b_k} \to S^{a_n}_{b_k}$,

$$(A\psi)(x) = F^{-1}[\varphi(\sigma)F[\psi](\sigma)](x),\ \forall\psi \in S^{a_n}_{b_k}.$$

Now consider the operator $\varphi(i\frac{d}{dx})$, where $\varphi \in \Theta^{b_n}_{a_k}$. Since $i\frac{d}{dx}$ is a self-adjoint operator in Hilbert space $L_2(\mathbb{R})$ with domain $D(i\frac{d}{dx}) = \{\psi \in L_2(\mathbb{R}) :\ \exists\psi' \in L_2(\mathbb{R})\}$, and φ is a real function, then $\varphi(i\frac{d}{dx})$ is also self-adjoint operator in $L_2(\mathbb{R})$ with a dense domain in $L_2(\mathbb{R})$. If E_λ, $\lambda \in \mathbb{R}$ is the spectral function of the operator $i\frac{d}{dx}$, then, due to the basic spectral theorem for self-adjoint operators we have

$$\varphi(i\frac{d}{dx})\psi = \int\limits_{-\infty}^{+\infty} \varphi(\lambda)dE_\lambda\psi,\ \forall\psi \in D(\varphi(i\frac{d}{dx})).$$

It is known (see, for example, [8]) that

$$E_\lambda\psi = \frac{1}{2\pi}\int\limits_{-\infty}^{\lambda} \left\{ \int\limits_{-\infty}^{+\infty} \psi(\tau)e^{i\sigma\tau}d\tau \right\} e^{-it\sigma}d\sigma.$$

Hence we get $dE_\lambda\psi = \frac{1}{2\pi}F[\psi](\lambda)e^{-it\lambda}d\lambda$. So

$$\varphi(i\frac{d}{dx})\psi = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} \varphi(\lambda)F[\psi](\lambda)e^{-it\lambda}d\lambda = F^{-1}[\varphi(\lambda)F[\psi](\lambda)],\ \forall\psi \in S^{a_n}_{b_k}.$$

Thus, in the space $S^{a_n}_{b_k}$ the pseudodifferential operator $A = F^{-1}[\varphi F]$ coincides with the operator $\varphi(i\frac{d}{dx})$, ie the pseudodifferential operator A can be understood as a constructive implementation of the operator $\varphi(i\frac{d}{dx})$.

As an example, consider the function $\varphi(\sigma) = (1+\sigma^2)^{\omega/2}$, $\sigma \in \mathbb{R}$, $\omega \in [1,2)$ is fixed parameter. We are directly convinced that the function $\varphi \in C^\infty(\mathbb{R})$ and has the properties:

$$\varphi(\sigma) = c_\varepsilon \exp\{\varepsilon|\sigma|^\omega\},\ \sigma \in \mathbb{R},\ c_\varepsilon = 2^{\omega/2}\max\{1, 1/\varepsilon\}$$

($\varepsilon > 0$ is arbitrary fixed parameter),

$$|D_\sigma^n \varphi(\sigma)| \leq c_0 B_0^n n!,\ n \in \mathbb{N},\ \sigma \in \mathbb{R},$$

where $c_0 = c_0(\omega) > 0$, $B_0 = B_0(\omega) > 0$. It follows that φ is a multiplier in the space $S^1_{1/\omega}$ (ie $\varphi \in \Theta^1_{1/\omega}$). Then, due to the basic spectral theorem for self-adjoint operators

$$\varphi(i\frac{d}{dx})\psi = (I + (i\frac{d}{dx})^2)^{\omega/2}\psi = (I - \frac{d^2}{dx^2})^{\omega/2}\psi = \int\limits_{-\infty}^{+\infty}(1+\sigma^2)^{\omega/2} dE_\lambda \psi =$$

$$= F^{-1}[(1+\sigma^2)^{\omega/2}F[\psi]],\ \forall \psi \in S_1^{1/\omega}.$$

The operator $(I - \frac{d^2}{dx^2})^{\omega/2}$ is called a fractional order differentiation operator, the constructive implementation of which in the space $S_1^{1/\omega}$ is a pseudodifferential operator constructed by function (symbol) $(1+\sigma^2)^{\omega/2}$, $\sigma \in \mathbb{R}$, - multiplier in space $S^1_{1/\omega}$. In particular, if $\omega = 1$, then in the space S_1^1 the operator $\sqrt{1-\Delta}$, $\Delta = d^2/dx^2$, coincides with the pseudodifferential operator $F^{-1}(1+\sigma^2)^{1/2}F]$.

4 THE SPACES OF GENERALIZED FUNCTION OF S' TYPE

The symbol $(S_{a_k}^{b_n})'$ will denote the space of all linear continuous functionals given on the main space $S_{a_k}^{b_n}$ with weak convergence, and its elements will be called generalized functions.

Regular generalized functions or regular functionals will be called linear continuous functionals, the action of which on the main function $\varphi \in S_{a_k}^{b_n}$ is given by the formula

$$\langle f, \varphi\rangle = \int\limits_{\mathbb{R}} f(x)\varphi(x)dx.$$

Each locally integrable on $\mathbb{R}$ function f that satisfies the condition

$$\forall \varepsilon > 0\ \exists c_\varepsilon > 0\ \forall x \in \mathbb{R}:\ |f(x)| \leq c_\varepsilon(\widetilde{\gamma}(\varepsilon x))^{-1} \tag{9}$$

generates a regular generalized function $F_f \in (S_{a_k}^{b_n})'$:

$$\langle F_f, \varphi\rangle = \int\limits_{\mathbb{R}} f(x)\varphi(x)dx,\ \forall \varphi \in S_{a_k}^{b_n}.$$

It is correct the statement: *if the locally integrable on $\mathbb{R}$ functions f and g, which satisfy the condition* (9)*, do not coincide on the set of Lebesgue positive measure, then there exists a function $\varphi_0 \in S_{a_k}^{b_n}$ such that $\langle f, \varphi_0\rangle \neq \langle g, \varphi_0\rangle$, ie $F_f \neq F_g$. Conversely, if $F_f \neq F_g$, then the* *functions f and g do not coincide on the set of Lebesgue positive measure.* The proof of this statement is similar to the proof of the corresponding statement from [9].

The formulated statement allows us to identify locally integrable functions on $\mathbb{R}$ that satisfy the condition (9), with the generalized functions generated by them from the space $(S_{a_k}^{b_n})'$. It follows from the properties of the Lebesgue integral that the embedding $S_{a_k}^{b_n} \ni f \to F_f \in (S_{a_k}^{b_n})'$ is continuous.

Since in the main space $S_{a_k}^{b_n}$ the operation of shift of the argument T_x is defined, the convolution of the generalized function $f \in (S_{a_k}^{b_n})'$ with the main function is given by the formula

$$(f * \varphi)(x) = \langle f_\xi, T_{-x}\check{\varphi}(\xi)\rangle = \langle f_\xi, \varphi(x-\xi)\rangle,\ \check{\varphi}(\xi) = \varphi(-\xi)$$

(here $\langle f_\xi, T_{-x}\check{\varphi}(\xi)\rangle$ denotes the action of the functional f on the main function $T_{-x}\check{\varphi}(\xi)$ as a function of the variable ξ). From the property of infinite differentiability of the argument shift operation in the space $S_{a_k}^{b_n}$ it follows that the convolution $f * \varphi$ is an ordinary infinitely differentiable function on $\mathbb{R}$.

Let $f \in (S_{a_k}^{b_n})'$. If $f * \varphi \in S_{a_k}^{b_n}$, $\forall \varphi \in S_{a_k}^{b_n}$ and from the relation $\varphi_\nu \to 0$ at $\nu \to \infty$ by space topology $S_{a_k}^{b_n}$ it follows that the convolution $f * \varphi_\nu \to 0$ at $\nu \to \infty$ by space topology $S_{a_k}^{b_n}$, then the functional f is called a convolutor in the space $S_{a_k}^{b_n}$. For example, δ is the Dirac function is a convolutor in each space $S_{a_k}^{b_n}$:

$$\forall \varphi \in S_{a_k}^{b_n}:\ (\delta * \varphi)(x) = \langle \delta_\xi, \varphi(x-\xi)\rangle = \varphi(x).$$

The Fourier transform of the generalized function $f \in (S_{a_k}^{b_n})'$ is denoted by the relation $\langle F[f], \varphi\rangle = \langle f, F[\varphi]\rangle$, $\forall \varphi \in S_{b_k}^{a_n}$. Hence we get that $F[f] \in (S_{b_k}^{a_n})'$, if $f \in (S_{a_k}^{b_n})'$. In this case, the operator $F : (S_{a_k}^{b_n})' \to (S_{b_k}^{a_n})'$ is continuous.

Theorem 2. *If the generalized function $f \in (S_{a_k}^{b_n})'$ is a convolutor in the space $S_{a_k}^{b_n}$, then for an arbitrary function $\varphi \in S_{a_k}^{b_n}$ the formula*

$$F[f * \varphi] = F[f] \cdot F[\varphi]$$

is correct.

Proof. According to the condition of the theorem, $f * \varphi \in S_{a_k}^{b_n}$, $\forall \varphi \in S_{a_k}^{b_n}$. Then, using the definition of the Fourier transform of generalized functions from the space $(S_{a_k}^{b_n})'$, as well as the definition of the convolution of a generalized function with the main one, we write the following relations:

$$\forall \psi \in S_{a_k}^{b_n} : \langle F[f * \varphi], \psi\rangle = \langle f * \varphi, F[\psi]\rangle = \int_{-\infty}^{+\infty} (f * \varphi)(x) F[\psi](x)dx =$$

$$= \int_{-\infty}^{+\infty} \langle f_\xi, \varphi(x-\xi)\rangle F[\psi](x)dx = \langle f_\xi, \int_{-\infty}^{+\infty} \varphi(x-\xi)F[\psi](x)dx\rangle \tag{10}$$

(here $f * \varphi$ is understood as a regular generalized function).

Let

$$\Im(\xi) := \int_{-\infty}^{+\infty} \varphi(x-\xi)F[\psi](x)dx.$$

Then, due to Fubini's theorem

$$\Im(\xi) := \int_{-\infty}^{+\infty} \varphi(x-\xi)(\int_{-\infty}^{+\infty} \psi(\sigma)e^{i\sigma x}d\sigma)dx = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \varphi(x-\xi)\psi(\sigma)e^{i\sigma x}d\sigma dx =$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \varphi(t)\psi(\sigma)e^{i\sigma t}e^{i\sigma\xi}d\sigma dt = \int_{-\infty}^{+\infty} \psi(\sigma)F[\varphi](\sigma)e^{i\sigma\xi}d\sigma = F[F[\varphi]\cdot\psi] \in S_{a_k}^{b_n}.$$

So,

$$\langle F[f*\varphi],\psi\rangle = \langle f, F[F[\varphi]\cdot\varphi]\rangle = \langle F[f], F[\varphi]\cdot\psi\rangle = \langle F[f]\cdot F[\varphi],\psi\rangle,\ \forall\psi\in S_{b_k}^{a_n}.$$

Hence we get equality $F[f*\varphi] = F[f]\cdot F[\varphi]$.

It remains to justify the correctness of the relationship (10). Define the notation

$$\Im_r(\xi) := \int_{-r}^{r} \psi(\sigma)F[\varphi](\sigma)e^{i\sigma\xi}d\sigma,\ r>0.$$

To prove (10) it suffices to establish that $\Im_r(\xi)\to\Im(\xi)$ for $r\to+\infty$ in the space $S_{a_k}^{b_n}$, ie $\alpha_r(\xi) := \Im(\xi)-\Im_r(\xi)\to 0$ for $r\to+\infty$ by space topology $S_{a_k}^{b_n}$. This means that: 1) the family of functions $\{\alpha_r^{(n)}(\xi),\ r>0\}$, $n\in\mathbb{Z}_+$, coincides to zero at $r\to+\infty$ evenly on each segment $[a,b]\subset\mathbb{R}$; 2) $|\alpha_r^{(n)}(\xi)|\le cB^n b_n\widetilde{\gamma}(a\xi)$, $\forall n\in\mathbb{Z}_+$,
where constants $c,\ B,\ a>0$ do not depend on r.

The integral

$$\int_{-\infty}^{+\infty} D_\xi^n(\psi(\sigma)F[\varphi](\sigma)e^{i\sigma\xi})d\sigma = \int_{-\infty}^{+\infty} (i\sigma)^n\psi(\sigma)F[\varphi](\sigma)e^{i\sigma\xi}d\sigma$$

coincides uniformly with respect to ξ, since

$$\forall\xi\in\mathbb{R} : |D_\xi^n(\psi(\sigma)F[\varphi](\sigma)e^{i\sigma\xi})| \le |\sigma^n\psi(\sigma)F[\varphi](\sigma)|,\ \sigma\in\mathbb{R},$$

$$\int_{-\infty}^{+\infty} |\sigma^n\psi(\sigma)F[\varphi](\sigma)|d\sigma < \infty$$

(since $\sigma^n\psi(\sigma)F[\varphi](\sigma)\in S_{b_k}^{a_n}$ for every $n\in\mathbb{Z}_+$). Then

$$|\alpha_r^{(n)}(\xi)| \le \int_{|\sigma|\ge r} |\sigma^n\psi(\sigma)F[\varphi](\sigma)|d\sigma \to 0,\ r\to\infty,$$

uniformly by ξ as the remainder of the convergent integral. Therefore, condition 1) is fulfilled. Let's check fulfillment of condition 2). Since

$$D_\xi^n \alpha_r(\xi) = D_\xi^n \mathfrak{I}(\xi) - D_\xi^n \mathfrak{I}_r(\xi),$$

then $|D_\xi^n \alpha_r(\xi)| \leq |D_\xi^n \mathfrak{I}(\xi)| + |D_\xi^n \mathfrak{I}_r(\xi)|$. Consider the functions

$$D_\xi^n \mathfrak{I}_{r,+}(\xi) = \max(D_\xi^n \mathfrak{I}_r(\xi), 0), D_\xi^n \mathfrak{I}_{r,-}(\xi) = -\min(D_\xi^n \mathfrak{I}_r(\xi), 0),$$

which are non-negative and consider that

$$|D_\xi^n \mathfrak{I}_r(\xi)| = D_\xi^n \mathfrak{I}_{r,+}(\xi) + D_\xi^n \mathfrak{I}_{r,-}(\xi) \leq 2|D_\xi^n \mathfrak{I}(\xi)|.$$

Then

$$|D_\xi^n \alpha_r(\xi)| \leq 3|D_\xi^n \mathfrak{I}(\xi)| = 3|D_\xi^n F[F[\varphi] \cdot \psi]|, \ \forall r > 0. \tag{11}$$

Since $F[F[\varphi] \cdot \psi] \in S_{a_k}^{b_n}$, $\forall \varphi \in S_{a_k}^{b_n}$, $\psi \in S_{b_k}^{a_n}$, then hence and from (11) follows the estimate

$$|D_\xi^n \alpha_r(\xi)| \leq cB^n b_n \widetilde{\gamma}(a\xi), \ n \in \mathbb{Z}_+, \ \xi \in \mathbb{R},$$

where constants $c, \ a, \ B > 0$ do not depend on r. Thus, condition 2) is satisfied. Theorem proved. □

Corollary 1. *If the generalized function f is a convolutor in the space $S_{a_k}^{b_n}$, then its Fourier transform is a multiplier in the space $S_{b_k}^{a_n}$.*

Proof. The mapping $(0, T] \ni t \to f_t(\cdot) \in S_{a_k}^{b_n}$ will be called an abstract function of the parameter t with values in the space $S_{a_k}^{b_n}$. Hereinafter, this mapping will be denoted by the symbol $f_t(\cdot)$, and let $f_t(\cdot)$ be the differential function of the parameter t, ie, the limits relation

$$\frac{f_{t+\Delta t}(\cdot) - f_t(\cdot)}{\Delta t} \to \frac{\partial}{\partial t} f_t(\cdot), \ \Delta t \to 0, \tag{12}$$

is performed in the sense of convergence in the topology of $S_{a_k}^{b_n}$ space. Then the formula

$$\frac{\partial}{\partial t}(f * f_t(\cdot)) = f * \frac{\partial}{\partial t} f_t(\cdot), \ \forall f \in (S_{a_k}^{b_n})'. \tag{13}$$

is correct. In fact, according to the definition of convolution, we have a generalized function with the main one

$$f * f_t(\xi) = \langle f_\xi, T_{-x} \check{f}_t(\xi) \rangle = \langle f_\xi, f_t(x - \xi) \rangle, \ \check{f}_t(x) = f_t(-x).$$

Then

$$\frac{\partial}{\partial t}(f * f_t(\cdot)) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}[f * f_{t+\Delta t}(\cdot) - f * f_t(\cdot)] = \lim_{\Delta t \to 0} \langle f_\xi, \frac{1}{\Delta t}[T_{-x} \check{f}_{t+\Delta t}(\xi) - T_{-x} \check{f}_t(\xi)] \rangle.$$

Given (12) we have that the limits relationship

$$\frac{1}{\Delta t}[T_{-x} \check{f}_{t+\Delta t}(\cdot) - T_{-x} \check{f}_t(\cdot)] \xrightarrow[\Delta t \to 0]{} \frac{\partial}{\partial t} T_{-x} \check{f}_t(\cdot)$$

is performed in the sense of convergence in the topology of space $S_{a_k}^{b_n}$. From here, taking into account the continuity of the functional f, we obtain

$$\frac{\partial}{\partial t}(f * f_t(\cdot)) = \langle f_\xi, \lim_{\Delta t \to 0} \frac{1}{\Delta t}[T_{-x}\check{f}_{t+\Delta t}(\xi) - T_{-x}\check{f}_t(\xi)]\rangle =$$

$$= \langle f_\xi, \frac{\partial}{\partial t} T_{-x}\check{f}_t(\xi)\rangle = \langle f_\xi, T_{-x}\frac{\partial}{\partial t}\check{f}_t(\xi)\rangle = f * \frac{\partial}{\partial t} f_t(\xi),$$

which had to be proved. □

5 The Cauchy problem

Let us set a problem: find the solution of the differential-operator equation

$$\frac{\partial u(t,x)}{\partial t} + \sqrt{I - \Delta}u(t,x) = 0, \ (t,x) \in (0,T] \times \mathbb{R} \equiv \Omega, \tag{14}$$

which satisfy the condition

$$u(t,\cdot)|_{t=0} = f, \ f \in (S_1^1, {}_*)'. \tag{15}$$

Here the symbol $(S_1^1, {}_*)'$ denotes the class of convolutors in the space S_1^1, $\Delta = \partial^2/\partial x^2$, operator $\sqrt{I-\Delta}$ is understood as a pseudodifferential operator in the space S_1^1, constructed by the function (symbol) $\varphi(\sigma) = (1+\sigma^2)^{1/2}$, $\sigma \in \mathbb{R}$ (see n.3), ie $\sqrt{I-\Delta}\psi(x) = F^{-1}[(1+\sigma^2)^{1/2}F[\psi]](x)$, $\forall \psi \in S_1^1$.

As the solution of the Cauchy problem (14), (15) we understand the function u which is differentiable by t $u(t,x)$ such that $u(t,\cdot) \in S_1^1$ for each $t \in (0,T]$, $u(t,x)$, $(t,x) \in \Omega$, satisfies the equation (14) (in the usual sense) and the initial condition (15) in the sense that $u(t,\cdot) \to f$ for $t \to +0$ in the space $(S_1^1)'$, ie

$$\langle u(t,\cdot), \psi\rangle \to \langle f, \psi\rangle, \ t \to +0,$$

for an arbitrary function $\psi \in S_1^1$ (here $u(t,\cdot)$ for each $t \in [0,T]$ is understood as a regular generalized function from the $(S_1^1)'$ space).

This formulation of the problem allows us to extend the class of initial functions in which the problem (14), (15) is correctly solvable, and the solution $u(t,x)$ is infinitely differentiable with respect to the variable x . For example, consider a function

$$f(x) = \begin{cases} \exp\{|x|^{-\alpha}\}, \ x \in [-1,1] \setminus \{0\}, \\ 0, \ |x| > 1, \end{cases}$$

where $\alpha > 0$ is a fixed parameter. This function has a feature of "exponential" type at the point $x = 0$ and allows regularization in the space $(S_1^\beta)'$, where $1 < \beta < 1 + 1/\alpha$ (see [10]) , ie f is a regular generalized function from the space $(S_1^\beta)' \subset (S_1^1)'$. The carrier of the generalized function f (*supp* f) consistent with the segment $[-1,1]$, ie f is a finite functional. Note also that each finite generalized function is a convolutor in the space S_1^1. This property follows from the general result related to the theory of perfect spaces (see [1, p. 173]): if Φ is a perfect space with a differential shift operation, then each finite functional is a convolutor in the space Φ. Finite generalized functions form a fairly broad class. In particular, each bounded closed set $F \subset \mathbb{R}$ is a carrier of some generalized function (see, for example, [4, p. 118]).

If we now return to the Cauchy problem (14), (15), we find the solution of the equation (14) by means of the Fourier transform. As a result, we find that the solution of the problem (14), (15) is given by the formula

$$u(t,x) = f * G(t,x) \equiv \langle f_\xi, G(t, x-\xi)\rangle, \ f \in (S_1^1, {}_*)'.$$

where

$$G(t,x) = F^{-1}[Q(t,\sigma)], \ Q(t,\sigma) = \exp\{-t(1+\sigma^2)^{1/2}\}, \ \sigma \in \mathbb{R}.$$

We are directly make shure that $Q(t,\cdot) \in S_1^1$ for each $t \in [0,T]$. Then $G(t,\cdot) = F^{-1}[Q(t,\cdot)]$ is also an element of the space S_1^1, since $F^{-1}[S_1^1] = S_1^1$. The function $G(t,\cdot)$ as abstract function of the parameter t with values in the space S_1^1, differentiable by t, then, according to the formula (13) we have

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial}{\partial t}(f * G(t,x)) = f * \frac{\partial G(t,x)}{\partial t}.$$

In addition, since the generalized function f is a convolutor in the space S_1^1, then, by Theorem 2,

$$\sqrt{I-\Delta}u(t,x) = F^{-1}[(1+\sigma^2)^{1/2}F[u(t,x)]] = F^{-1}[(1+\sigma^2)^{1/2}F[f * G(t,x)]] =$$
$$= F^{-1}[(1+\sigma^2)^{1/2}F[f]F[G]] = F^{-1}[(1+\sigma^2)^{1/2}F[f]Q(t,\sigma)] = -F^{-1}[\frac{\partial}{\partial t}Q(t,\sigma)F[f]] =$$
$$= -F^{-1}[F[\frac{\partial}{\partial t}G]F[f]] = -F^{-1}[F[f * \frac{\partial G}{\partial t}]] = -f * \frac{\partial G(t,x)}{\partial t}.$$

Hence we get that the function $u(t,x)$, $(t,x) \in \Omega$, satisfies the equation (14). In addition, $Q(t,\cdot) \to 1$ for $t \to +0$ in the space $(S_1^1)'$. Hence we get $G(t,x) = F^{-1}[Q(t,\sigma)] \to F^{-1}[1] = \delta$ for $t \to +0$ in space $(S_1^1)'$ (here δ is Dirac delta-function).

Since $F[f]$ is a multiplier in the space S_1^1, if $f \in (S_1^1, {}_*)'$ (see Corollary 1), then $F[f]\psi \in S_1^1$ for $\forall \psi \in S_1^1$. Then

$$\langle F[u(t,\cdot)], \psi\rangle = \langle F[f * G(t,\cdot)], \psi\rangle = \langle F[f]F[G(t,\cdot)], \psi\rangle = \langle F[f]Q(t,\cdot), \psi\rangle =$$
$$= \langle Q(t,\cdot), F[f]\psi\rangle \xrightarrow[t\to+0]{} \langle 1, F[f]\psi\rangle = \langle F[f], \psi\rangle, \ \forall \psi \in S_1^1,$$

that is, $F[u(t,\cdot)] \to F[f]$ for $t \to +0$ in the space $(S_1^1)'$. It follows that $u(t,\cdot) \to f$ for $t \to +0$ in the space $(S_1^1)'$, ie $u(t,x)$ satisfies the condition (15) in the specified sense. We can also prove that the problem (14), (15) has a single solution. Thus, the Cauchy problem (14), (15) is correctly solvable, the solution is given by the formula $u(t,x) = f * G(t,x)$, $(t,x) \in \Omega$, with $u(t,\cdot) \in S_1^1$ for each $t \in [0,T]$.

Similar results occur in the case of the equation

$$\frac{\partial u(t,x)}{\partial t} + (I - \frac{\partial^2}{\partial x^2})^{\omega/2}u(t,x) = 0, \ (t,x) \in \Omega, \ \omega \in [1,2). \quad (16)$$

The Cauchy problem for the equation (16) is correctly solvable if the initial function f is a convolutor in the space $S_1^{1/\omega}$, ie $f \in (S_{1,*}^{1/\omega})'$, and the solution is given by the formula

$$u(t,x) = f * G(t,x),$$

where $G(t,x) = F^{-1}[\ exp\{-t(1+\sigma^2)^{\omega/2}\}]$.

The considered (schematically) Cauchy problem can be called a model. According to the given scheme it is possible to investigate Cauchy problems and nonlocal in time problems for equations of more general kind.

Conclusion

The topological structure of generalized spaces of type S, properties of operations important for mathematical analysis (argument shift, multiplication by an independent variable, differentiation), the class of multipliers in such spaces are described. The question of quasi-analyticity (non-quasi-analyticity) of generalized spaces of S type is studied. We find a condition under which in generalized spaces of type S are defined, are linear and continuous pseudodifferential operators constructed on certain symbols, such operators are understood as a constructive implementation of the operators $\varphi(i\frac{d}{dx})$ y such spaces.

These results can be used in the study of the Cauchy problem and nonlocal time problems for differential operator equations of the form $\partial u(t,x)/\partial t = \varphi(i\partial/\partial x)u(t,x)$, $(t,x) \in \Omega$, with an initial function that is an element of the space of generalized functions of type ultradistributions (S' type).

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Городецький В.В., Колісник Р.С., Шевчук Н.М. *Узагальнені простори типу S та S'* // Буковинський матем. журнал — 2023. — Т.11, №1. — С. 7–25.

У роботі досліджено топологічну структуру узагальнених просторів типу S, властивості операцій, важливих для математичного аналізу (зсуву аргумента, множення на незалежну змінну, диференціювання), описано клас мультиплікаторів у таких просторах. Вивчено питання про квазіаналітичність (неквазіаналітичність) узагальнених просторів типу S. Знайдено умову, при виконанні якої в узагальнених просторах типу S визначені, є лінійними і неперервними псевдодиференціальні оператори, побудовані за певними символами, такі оператори розуміємо як конструктивну реалізацію операторів $\varphi(i\frac{d}{dx})$ у таких просторах.

Наведені результати можна використовувати при дослідженні задачі Коші та нелокальних за часом задач для диференціально-операторних рівнянь вигляду $\frac{\partial u(t,x)}{\partial t} = \varphi(\frac{i\partial}{\partial x})u(t,x),\ (t,x) \in \Omega$ з початковою функцією, яка є елементом простору узагальнених функцій типу ультрарозподілів (типу S).